# CALCULUS AMONG FRIENDS 

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## CHAPTER 1: THE DEF OF DIFF

## Warm-up:

Just a few questions, along with their answers, for you to ponder:
When you ride in a car, does the speedometer always read the same? Answer: No.
How long does it usually stay put? Answer: Not very.
Could it possibly read 60 mph at 2:00 and 80 a minute later? Answer: Yes.
What about half a minute later? Answer: Yes.

What about half a second? Answer: Theoretically, yes.
There is no end to how frenetic a speedometer can, theoretically, be. Speed, or velocity, can be very inconstant. 60 mph at $2: 00$ says nothing about how many mph at any other time. No velocity is guaranteed to last any amount of time at all. Perhaps you've heard that the mysterious "derivatives" of Calculus represent "instantaneous velocity" or "instantaneous change".

So here's one more warm-up question: What does 60 mph mean, anyway?
And here are some possible answers:

1) if you keep going at that same speed for an hour, you'll have traveled 60 miles by the end of that hour
2) if you average that rate over the hour, you'll have traveled 60 miles by the end of that hour
3) (the calculus meaning) that 60 is the limit, over shorter and shorter time-intervals, each starting at 2:00, of the distance traveled divided by the time taken to do the traveling.

Instantaneous velocity is what the speedometer says, and it's one example of a derivative. Other examples come from other applications, other things that change. Rate of change is the bottom line, so in most of what you'll see in this book, units like mph and hours are omitted, and everything is composed of numbers and variables - ready to be applied to many situations, motion of cars, particles, and rocket ships being only an example.

## The Case of the Missing Speedometer

Here's a typical calculus-type hypothetical scenario: What if your car had no speedometer, but did have an odometer (device for measuring distance already traveled)?

And what if you were asked for the car's speed at, say, 2:00 PM?
Could you find it, using only the odometer and your watch (or cell phone)?

Remember, no speedometer, so you can't just read off the speed. (Hint: Recall the noncalculus formula, $\mathrm{r}=\frac{d}{t}$, for the average speed, where $\mathrm{d}=$ the distance traveled and $\mathrm{t}=$ the length of time of the traveling.)

Let's first approximate. That is, let's give what's known in math as a "first approximation" (before we get a better approximation). We might say:

Speed at 2:00 PM equals approximately

```
distance - travelled - between - \(2: 00\) - and - \(2: 05\)
    5
```

This is a good place to begin using a little mathematical notation. It will make things take up less space and also be more visual. So ~ will be short for "equals approximately", and $\mathrm{D}(\mathrm{t})$ will be short for the distance traveled by t o'clock, in our case, for now, $\mathrm{t}=2: 05$. Thus in our shorthand notation we have:

Speed at $2: 00 \sim \frac{D(2: 05)-D(2: 00)}{5}$

This formula gives our "first approximation" to the speed at 2:00 PM. We didn't use any speedometer. Instead we (hypothetically) used our odometer (for the numerator) and our watch (for the denominator).

Can we get a better approximation?
Yes, by taking a shorter time interval than 5 minutes - that is, by reading our odometer at a time closer to 2:00 than 2:05 - perhaps 2:02. (The approximation is likely to be better because there is less time for the speed to vary in.) In our shorthand mathematical notation our "second approximation" to the speed at 2:00 might be written:

Speed at $2: 00 \sim \frac{D(2: 02)-D(2: 00)}{2}$

2:05 has been replaced by $2: 02$. Perhaps we now see that we could get a third approximation. For example:

Speed at $2: 00 \sim \frac{D(2: 01)-D(2: 00)}{1}$

We could perhaps get boring and do a "fourth approximation". We'd need to use a fraction of a minute, and the denominator would be $1 / 2$. However, instead of doing that, let's make a giant leap.

In general, and definitely less boring, we'd could let $\Delta t$ stand for the length of the time interval - meaning how much time has elapsed since 2:00. $\Delta \mathrm{t}$ is read "delta t ", for the Greek letter delta, and delta anything indicates the change in anything. More about that notation later. Our approximation (corresponding to whatever $\Delta t$ we choose) would be, same ideas as before:

Speed at $2: 00 \sim \frac{D(2: 00+\Delta t)-D(2: 00)}{\Delta t}$
As we have seen, the smaller the time interval $\Delta t$, the closer our approximation is likely to be.

How close can we get? What's the bottom line here? In calculus, the way to phrase that question is: What's the limit? So let's just write "limit", or rather "lim", our math shorthand for "limit":

Speed at $2: 00=\lim _{\Delta t \rightarrow 0} \frac{D(2: 00+\Delta t)-D(2: 00)}{\Delta t}$
And yes, that's an equal sign, not an approximation sign, because we're talking about the limit of the "average speeds", and the little arrow means "gets closer and closer to" or "approaches" (but never reaches).

There is a more mathematically rigorous way of defining "lim", along with that little arrow, but let's be intuitive for now. We know what we mean!

So, that "lim expression" is exactly what the speed AT 2:00 is. It's what, theoretically, the speedometer would say, or rather it's what the speedometer MEANS. (If we were being practical, we wouldn't worry about the lim. We'd just get reasonably close, perhaps by taking $\Delta \mathrm{t}=1$ (1 minute).

At this point some people might ask, "Why can't we just take $\Delta t=0$ - that is, NO time elapsing between 2:00 and 2:00? Why do we have to worry about the limit?" The answer is that it wouldn't work; it wouldn't get us anywhere. Here's what would happen:

Speed at $2: 00=\frac{D(2: 00)-D(2: 00)}{0}=\frac{0}{0}--$ ?????????????????????????????????????????
$\frac{0}{0}$ is, as it's often put, "undefinted". It could be any number, since 0 times any number is 0 . Thus this gives us no info.

Now, while the above formula, without knowing anything else (namely, calculus) wouldn't serve to exactly solve our missing speedometer problem (since it involves an infinitude of numbers), it is extremely profound and important, and useful once we learn how to handle limits - that is, once we learn some calculus. (We'd have to know $\mathrm{D}(\mathrm{t})$ as a function of time t , and we'd have to plug in some numbers.)

This limit formula provides the essence of calculus. It's the definition of the derivative of the function D , or $\mathrm{D}(\mathrm{t})$, at the "time" 2:00, and again, it's what the speedometer means by the speed "at 2:00".

Now - another giant leap -- reflect that other things change besides distance. In other applications of calculus, not only the distance traveled by cars, rockets, and particles, but electrical charge, too, can change, populations of countries and of bacteria cultures can change, and speed itself can change (that's what we mean by acceleration - change in speed). That is, the function D - or whatever you choose to call it -- doesn't have to be distance.

And there's more. t - or whatever you chose to call it - doesn't have to be time. For example, in business it could be the quantity produced. (Profit and cost are both functions of the quantity produced, called "level of production".) There are other examples of what D and t could be, other applications of calculus besides speed.

Perhaps you now have some idea of what "function of a variable" means (or perhaps you already knew). But here's a quick summary, an actual definition of "function": A function $f$ is a rule which assigns, to every number (or sometimes only certain numbers) another number $f(x) . f(x)$ is called the value of $f$ at $x$. Functions are treated more fully in Chapter 2. And limits are treated more fully in Chapter 3. If you like, you can take a peek at those later sections; you'll understand them!

A short, rather visual, almost-correct description of "function" is "anything with x in it" (or t , or any other "independent variable"). The exception in this almost-correct description is constant functions - such as 2 . The function 2 assigns, to every number, the value 2. (You might remember that the graph of this function 2 is a horizontal straight line 2 units above the x -axis.)

Let us now do the main thing of calculus, which is define what "derivative" means. It will look a lot like the limit formula from "the case of the missing speedometer"; in fact, it a generalization of that formula. T is replaced by $\mathrm{x}, \Delta \mathrm{t}$ by $\Delta \mathrm{x}, \mathrm{D}$ by f , and $2: 00$ by $\mathrm{x}_{0}$ :

DEFINTION: Given a function $\mathrm{f}(\mathrm{x})$ and a point (number) $\mathrm{x}_{0}$, the derivative of fat $x_{0}$
is defined to be: $\lim _{\Delta x \rightarrow 0} \frac{f\left(x_{o}+\Delta x\right)-f\left(x_{o}\right)}{\Delta x}$
NOTATION: This derivative is denoted, for short, by $\mathrm{f}^{\prime}\left(\mathrm{x}_{o}\right)$ and read " f -prime of x naught". Sometimes, also, $\Delta \mathrm{x}$ is replaced by h . h takes less time to write when we actually compute derivatives via this limit formula (and we'll be doing that very soon). So the derivative of f at $\mathrm{x}_{o}$ could also, equivalently, be defined as:

$$
\lim _{h \rightarrow o} \frac{f\left(x_{o}+h\right)-f\left(x_{o}\right)}{h}
$$

To recapitulate and clarify: $\mathrm{x}_{o}$ is the abstraction of 2:00 and $\mathrm{f}(\mathrm{x})$ is the abstraction of the distance function $\mathrm{D}(\mathrm{t})$. Also, $\mathrm{f}^{\prime}\left(\mathrm{x}_{o}\right)$ is always a number, and can always be found when f and $\mathrm{x}_{o}$ are given.

MORE NOTATION: We could also, in the formula, use $\mathrm{x}-\mathrm{x}_{o}$ instead of h . Then we'd have $\mathrm{x}=\mathrm{x}_{o}+\mathrm{h}$, and the formula would become:

$$
\mathrm{f}^{\prime}\left(\mathrm{x}_{o}\right)=\lim _{x \rightarrow x_{o}} \frac{f(x)-f\left(x_{o}\right)}{x-x_{o}}
$$

Either way the derivative is the limit of the difference in the function values divided by the difference in the independent variable x . As we move along, we'll see the formula both ways. The beauty of this new way is that both numerator and denominator are seen as differences.

THE LAST PIECE OF NOTATION (at least for this section): Since we've been using the delta notation $\Delta x$ to represent the change in x , we can analogously use it to represent the corresponding change in the function f - that is, $\Delta \mathrm{f}$. But mathematicians don't! Instead they, like anybody who remembers analytic geometry, use $y$ to designate the function value (since $y=f(x)$ when we graph $f(x)$ ). Thus we sometimes write our derivative formula yet another way:

$$
\mathrm{f}^{\prime}\left(\mathrm{x}_{o}\right)=\lim _{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}
$$

And each $\frac{\Delta y}{\Delta x}$, without the limit sign, is called a difference-quotient, because of the subtraction and division involved. Thus the derivative is a limit of different-quotients. (Actually, "differences-quotient" might be a better term, since there are actually two differences.) The difference-quotient is the same thing as the average rate of change.)

We also sometimes write the derivative as $\frac{d y}{d x}\left(x_{o}\right)$ or just-plain $\frac{d y}{d x}$. And we read it "d-y-$d-x "$.

Remember that dx and dy are NOT separate quantities, only the ENTIRE symbol $\frac{d y}{d x}$ means anything. (Later on we'll take liberties, "pretend" we think dx and dy are separate quantities, and do a kind of algebra with them accordingly. This is permissible because, roughly, while not a quotient, $\frac{d y}{d x}$ is the limit of quotients. Keep in mind, however, that d is not a separate quantity.)

Summary so far: Given a function $\mathrm{f}(\mathrm{x})$ and a point (number) $\mathrm{x}_{o}$, the derivative $\mathrm{f}^{\prime}\left(\mathrm{x}_{o}\right)$ of $\mathrm{f}(\mathrm{x})$ at $\mathrm{x}_{o}$ is defined to be the number:

$$
\lim _{h \rightarrow 0} \frac{f\left(x_{o}+h\right)-f\left(x_{o}\right)}{h}
$$

which is also equal to the number:

$$
\lim _{x \rightarrow x_{o}} \frac{f(x)-f\left(x_{o}\right)}{x-x_{o}}
$$

This derivative gives the instantaneous change in the function $\mathrm{f}(\mathrm{x})$ at the point $\mathrm{x}_{o}$

## Shall We Diff?

Look again at our definition (the one with the h in it). Reflect that $\mathrm{x}_{o}$ is any arbitrary point, so why not save time and space and just call it x ? We'd now have:

$$
\mathrm{f}^{\prime}(\mathrm{x})=\lim _{h \rightarrow 0} \frac{f(x+h)-f(x)}{h}
$$

Now note something interesting and important: $\mathrm{f}^{\prime}(\mathrm{x})$ is also a function of $x$. $\left(\mathrm{f}^{\prime}(\mathrm{x})\right.$ does not depend on $h$, since $h$ literally disappears when we let $h \rightarrow 0$.) $f^{\prime}(x)$ is another function, in general different from $f(x)$. So every function $f(x)$ (whether it's the speed of a car or the level of production in a factory) might (if the above limit exists; it doesn't always) give rise to a derivative function $f^{\prime}(x)$. (Sometimes even, we write $f$ and $f^{\prime}$, without the x ; like all shorthand notation, it makes things simpler and more visual.)

So, a typical calculus problem could be coached as: Given $f(x)$, find $f^{\prime}(x)$. We're now in a position to do many such problems! Without further ado, let's calculate $f^{\prime}(x)$ when $f(x)=$ 2 x . (Later in this section we'll do $3 \mathrm{x}, \mathrm{kx}$ for any number $\mathrm{k}, \mathrm{x}^{2}$, and $\mathrm{x}^{3}$.)

So if $f(x)=2 x$, we have:

$$
\begin{aligned}
\mathrm{f}^{\prime}(\mathrm{x}) & =\lim _{h \rightarrow 0} \frac{f(x+h)-f(x)}{h}, \text { by our definition above of derivative } \\
& =\lim _{h \rightarrow 0} \frac{2(x+h)-2 x}{h}, \text { since } \mathrm{f}(\mathrm{x})=2 \mathrm{x}-\text { and therefore } \mathrm{f}(\mathrm{x}+\mathrm{h})=2(\mathrm{x}+\mathrm{h}) \\
& =\lim _{h \rightarrow 0} \frac{2 x+2 h-2 x}{h}, \text { by algebra on the numerator (the distributive law) } \\
& \left.=\lim _{h \rightarrow 0} \frac{2 h}{h}, \text { by more algebra on the numerator (canceling the } 2 \mathrm{x}\right)
\end{aligned}
$$

$$
=\lim _{h \rightarrow 0} 2, \text { by canceling the h's }
$$

$=2$, because, roughly, " 2 doesn't care what h does". (There's no h in 2. .)
So - the derivative of 2 x is 2 . A similar calculation, with 3 instead of 2 , shows that the derivative of 3 x is 3 . Giant leap (maybe not quite so giant...): the derivative of kx , where k is any number, is k .

This last jibes with our original distance/velocity example in the first section, in the case of constant velocity. If instead of $x$ we use $t$, instead of $f$ we use $D$, and instead of $k$ we use r (for rate, or velocity), then $\mathrm{D}(\mathrm{t})=\mathrm{rt}$, and it isn't surprising that the derivative (speed) is r , since the derivative is the speed. So if we call the rate k rather than r , the derivative/speed of a car whose distance already traveled is kt is k .

As we've perhaps begun to see, in computing what I like to call "deriv's", lim's are often nothing to be afraid of. Basically we first deal, using the algebra we all know and love, with everything except the $\lim$ and then, at the very end, own up to lim, at which point it's much clearer what to do. (Take another look at the last two lines in the calculation above. Later, when we deal with other functions, we'll see that calculating deriv's is often more complicated, but that's later. And we can still get pretty far by first procrastinating dealing with that lim.)

Let next try $f(x)=x^{2}$. (And perhaps, as we do so, you might like to ask yourselves how the computation will generalize to any power of $x$ ).

$$
\begin{aligned}
& \mathrm{f}^{\prime}(\mathrm{x})=\lim _{h \rightarrow 0} \frac{f(x+h)-f(x)}{h}, \text { by the definition of deriv (It always starts out } \\
& \text { this way.) } \\
&=\lim _{h \rightarrow 0} \frac{(x+h)^{2}-x^{2}}{h}, \text { replacing f everywhere (both places) with the } \\
& \text { squaring function } \\
&=\lim _{h \rightarrow 0} \frac{\left(x^{2}+2 x h+h^{2}\right)-x^{2}}{h}, \text { by FOILing in the numerator } \\
&=\lim _{h \rightarrow 0} \frac{2 x h+h^{2}}{h}, \text { by canceling out the } \mathrm{x}^{2} \text { in the numerator }
\end{aligned}
$$

$=\lim _{h \rightarrow 0}(2 x+h)$, by canceling (one) $h$, in both num. and denom.
$=2 \mathrm{x}$ - because, first, 2 x doesn't care what h does (any more than 2 did, in the previous example; that is, neither 2 not $2 x$ have an $h$ in it) and, second, h surely approaches 0 as $h$ approaches 0 ! (both simple, but profound and important, concepts)

So - the deriv of $\mathrm{x}^{2}$ is 2 x . (Back to our missing speedometer, that means - hypothetical as this might seem - that, replacing $x$ by time $t$, if $t^{2}$ is the distance traveled by our car in $t$ hours, then 2 t will be the speed at time t (in mph ).

Now - shall we diff $x^{3}$ ? (Perhaps ask yourself, how much more complicated will this be than diff-ing $x^{2}$ in the previous example? Perhaps, even, you'd like to try it on your own, then check to see whether you got the same answer as appears below.) For each step of the calculation, a reason will be supplied only when it differs from the previous example - in order to emphasize the progression of ideas.

Setting $\mathrm{f}(\mathrm{x})=\mathrm{x}^{3}$, we have:

$$
\begin{aligned}
\mathrm{f}^{\prime}(\mathrm{x}) & =\lim _{h \rightarrow 0} \frac{f(x+h)-f(x)}{h} \\
& =\lim _{h \rightarrow 0} \frac{(x+h)^{3}-x^{3}}{h} \\
& =\lim _{h \rightarrow 0} \frac{\left(x^{3}+3 x^{2} h+3 x h^{2}+h^{3}\right)-x^{3}}{h}, \text { by FOILing }(\mathrm{x}+\mathrm{h})^{3} \text { in the numerator } \\
& =\lim _{h \rightarrow 0} \frac{3 x^{2} h+3 x h^{2}+h^{3}}{h}, \text { by canceling the } \mathrm{x}^{3} \text { in the numerator } \\
& =\lim _{h \rightarrow 0}\left(3 x^{2}+3 x h+h^{2}\right)
\end{aligned}
$$

$$
=3 x^{2}
$$

So - the deriv of $x^{3}$ is $3 x^{2}$-- whether we're talking about cars, production level, or anything that's a function of something else (whether or not it's meaningful to ask about the instantaneous rate of change.)

We'll soon calculate deriv's of other functions. First, it's important to realize some good news: Once we've gotten these formulas, we won't have to go through the limit process again to find deriv's. That's what formulas are for! We'll be able to use these formulas to develop other formulas, as well as to solve problems. The limit formula is, in a sense, a last-ditch attempt to calculate some deriv's. It's a formula to use when there's no other formula - like Mary Poppins; "supercalifragisticexpioladosis" is "a word to use when there's no other word"!)

At this point, more terminology and notation are in order. This is GOOD news! It means there are better way to express the formulas we just found (as well as others). Here are ways to express the fact that the deriv of $x^{2}$ is $2 x$ :

$$
\begin{aligned}
& \frac{d\left(x^{2}\right)}{d x}=2 \mathrm{x} \\
& \frac{d}{d x}\left(x^{2}\right)=2 \mathrm{x}
\end{aligned}
$$

Also, if we don't know what the function is, and are writing about some arbitrary function $f(x)$ - to express some formula or fact about derivatives, and need some convenient notation - the following are all equivalent symbols for the same thing, namely the deriv of $\mathrm{f}(\mathrm{x})$ :

$$
\mathrm{f}^{\prime}(\mathrm{x})=\mathrm{f}^{\prime}=\frac{d f(x)}{d x}=\frac{d f}{d x}=\frac{d}{d x} f(x)=\frac{d}{d x} f
$$

All six mean the same thing. (Reminder: Later when we connect the idea of deriv's with analytic geometry, we'll sometimes be using y instead of $f$. Also, $f$ ' is read " $f$-prime" and $\frac{d y}{d x}$ is read "d-y-d-x".) You don't have to memorize all these symbols; you just need to be
able to recognize them. Later on you'll see that sometimes some symbols are more convenient than others, and why.

The process of finding the deriv of a given function is called differentiating or differentiation. (I just say "diff-ing", as you've seen.) Any function which can be diff'ed is said to be differentiable ("diff-able"). More precisely, if $\mathrm{f}^{\prime}\left(\mathrm{x}_{o}\right)$ exists, then f is said to be differentiable at $x_{o}$. As we've seen, $\mathrm{kx}, \mathrm{x}^{2}$, and $\mathrm{x}^{3}$ are all differentiable at all points, but we'll soon see functions (such as $\sqrt{x}$ ) which are diff-able at some points but not others.

Summary of this section:
We can use the limit formula to calculate derivatives of given functions. Formulas we have found so far are:

$$
\begin{aligned}
& \frac{d}{d x}(k x)=k \\
& \frac{d}{d x}\left(x^{2}\right)=2 x \\
& \frac{d}{d x}\left(x^{3}\right)=3 x^{2}
\end{aligned}
$$

## EXERCISES FOR CHAPTER 1

Use the limit formula, plus the three formulas at the end of the chapter, to find the deriv's of the following functions:
1.1) 5 x
1.2) $5 x^{2}$
1.3) $5 x^{3}$
1.4) $x$
1.5) -x
1.6) $\mathrm{x}^{4}$
1.7) $5 x^{4}$
1.8) $\frac{1}{x}$
1.9) 1
1.10) 5
1.11) 0

The following exercises are extra-credit, and constitute "sneak previews". This is an opportunity to learn by doing, to get some heads-up, and to have fun! Remember that you already have the deriv's of $\mathrm{kx}, \mathrm{x}^{2}$, and $\mathrm{x}^{3}$.
1.12) If a car doesn't move at all, what is its velocity?
1.13) Using the limit formula, prove that, if a function $f(x)$ is strictly increasing (that is, whenever $\mathrm{a}<\mathrm{b}$ we have $\mathrm{f}(\mathrm{a})<\mathrm{f}(\mathrm{b})$ ), then $\mathrm{f}^{\prime}(\mathrm{x})$ is non-negative for all x .
1.14) If a car is moving forward at all times, prove that its spend can't be negative. What happens if the car is moving backward?
1.15) What do you think it would mean for a function to be strictly decreasing? Prove that, if a function is strictly decreasing, its deriv is always non-positive.
1.16) Try to guess a general pattern for diff-ing any power of $x$.
1.17) Since the deriv of any function $f(x)$ is another function $f^{\prime}(x)$, we could diff again that is, we could diff the new function $f^{\prime}(x)$. This "deriv of the deriv" is called 'the second deriv", denoted by $f^{\prime}$ ' $(x)$ and read " $f$-double-prime". What is the second deriv of $x^{2}$ ? What is the second deriv of $x^{3}$ ? What is the second deriv of $x$ ?
1.18) Similarly, we could define third, fourth, etc., deriv's, of any function. What is the third deriv of $x^{2}$ ? What about is the fourth deriv of $x^{3}$ ?
1.19) Limits without deriv's:

Find the following limits:

$$
\begin{aligned}
& \lim _{x \rightarrow 0}(3+x) \\
& \lim _{x \rightarrow 1}(3+x) \\
& \lim _{x \rightarrow 1}(3+2 x) \\
& \lim _{x \rightarrow 1} \frac{\left(x^{2}-1\right)}{x-1}
\end{aligned}
$$

## CHAPTER 2: MAY I HAVE THE NEXT DIFF?

## Warm-up

You now know what diff-ing means, and you know how to diff a handful of functions constants, $x, x^{2}$, and $x^{3}$, If you did some of the later Exercises from the last chapter, you learned how to diff even higher powers of x. And if you didn't do those Exercises, you're about to learn how. The first section of this chapter will tell you.

But there are other functions besides powers - such as polynomials (linear combinations of powers), powers of polynomials, and products of powers of polynomials. And functions can be combined to make other functions. Thus there's a lot more diff-ing to do! This chapter will do only some of it. In Chapters 6 and 7 we'll take care of more of it, and Chapter 13 will, among other things, show you how to diff a rather large class of functions - exotic functions which you probably haven't yet heard of.

Let's begin with powers of $\mathrm{x}-$ all powers of x .

## The Powers That Be (The Power Rule)

Before getting a feel for why The Power Rule holds, you should know what it is:

The Power Rule: Let $n$ be any number. Then the derivative of the function $x^{n}$ is the function $n x^{n-1}$.

You can check this rule/formula for $\mathrm{n}=1,2$, and 3 . You'll see that it jibes with what you already know from Chapter 1. But it's true for all n . Let's see why, starting with $\mathrm{n}=4$. We're well equipped to do this, since we have everything in place, meaning the limit definition of diff-ing and some practice using it.

What happens when we begin to diff $x^{4}$ ? Well, something very similar to when we begin to diff $\mathrm{x}^{2}$ and $\mathrm{x}^{3}$. The only snag: FOILing $(\mathrm{x}+\mathrm{h})^{4}$ takes longer than FOILing $(\mathrm{x}+\mathrm{h})^{2}$ and $(\mathrm{x}+\mathrm{h})^{3}$. Good news, though: It turns out that we don't have to do all the FOILing. Watch what happens:

First: $\frac{d}{d x} x^{4}=\lim _{h \rightarrow 0} \frac{(x+h)^{4}-x^{4}}{h}$
And now, as we knew we would, we have to think about $(x+h)^{4}$. Well, written out the long way, this is equal to $(x+h)(x+h)(x+h)(x+h)$. When we FOIL that out, one term is $x^{4}$ (from multiplying together the four $x$ 's), and this, in the limit definition directly above, will cancel with the $-x^{4}$ on the extreme right. So in the numerator of the limit definition, we have to deal only with the other terms in $(x+h)^{4}=$ $(x+h)(x+h)(x+h)(x+h)$.

Let's next concentrate on another term. (Recall, from the diff-ing of $x^{2}$ and $x^{3}$ in Chapter 1, the other terms wound up not contributing anything to our final "answer" meaning once we've divided by $h$ and taken the limit) The term formed by multiplying three x 's and one h , in $(\mathrm{x}+\mathrm{h})(\mathrm{x}+\mathrm{h})(\mathrm{x}+\mathrm{h})(\mathrm{x}+\mathrm{h})$, is $\mathrm{Ax}^{3} \mathrm{~h}$, for some number A. (This is because $x$ gets used three times and $h$ gets used only once - but remember, it's not always the same $h$ that gets "chosen".)

What is that number A? Well, in how many ways can we get three x 's and one h ? The answer is 4 , because the one and only $h$ can come from either of the four ( $x+h$ )'s. Thus, another term in the expansion of $(x+h)(x+h)(x+h)(x+h)$. is $4 x^{3} h$.

Dealing with the other terms terms out to be a piece o' cake. Here's what happens: $(x+h)(x+h)(x+h)(x+h)=x^{4}+4 x^{3} h+B x^{2} h^{2}+C x h^{3}+h^{4}$, for some numbers $B$ and C. (We have only one $h^{4}$ for the same reason as there's only one $x^{4}$ ) We don't know what B and C are, but it doesn't matter. Here's what happens: Continuing with the
computation of the deriv of $\mathrm{x}^{4}$, using the limit definition, keeping in mind that B and C are numbers, we have:

$$
\begin{aligned}
& \frac{d}{d x} x^{4}=\lim _{h \rightarrow 0} \frac{(x+h)^{4}-x^{4}}{h} \\
&=\lim _{h \rightarrow 0} \frac{\left(x^{4}+4 x^{3} h+B x^{2} h^{2}+C x h^{3}+h^{4}\right)-x^{4}}{h} \\
&=\lim _{h \rightarrow 0} \frac{4 x^{3} h+B x^{2} h^{2}+C x h^{3}+h^{4}}{h}, \text { since the } \mathrm{x}^{4} \text { cancels } \\
&=\lim _{h \rightarrow 0}\left(4 x^{3}+B x^{2} h+C x h^{2}+h^{3)}, \text { by dividing num and denom by } \mathrm{h}\right. \\
&=4 \mathrm{x}^{3}-- \text { Yep! Most of the expressions in parentheses poof, because as } \\
& \text { h approaches } 0, \text { so do } \mathrm{h}^{2} \text { and } \mathrm{h}^{3} .
\end{aligned}
$$

Again, that happens regardless of what $B$ and $C$ are. So we just found that the deriv of $x^{4}$ is $4 x^{3}$, which jibes with the Power Rule stated at the beginning of this section. The next task is to generalize that computation - not only for $\mathrm{n}=4$, but for any n .
Here's the quick computation - that is, here's the quick proof of the Power Rule:

$$
\begin{aligned}
\frac{d x}{x} x^{n} & =\lim _{h \rightarrow 0} \frac{(x+h)^{n}-x^{n}}{h} \\
& =\lim _{h \rightarrow 0} \frac{x^{n}+n x^{n-1} h+B x^{n-2} h^{2}+C x^{n-3} h^{3}+\ldots-x^{n}}{h}
\end{aligned}
$$

Here the second term in the numerator is $\mathrm{nx}{ }^{n-1} \mathrm{~h}$ for the same reason that we had $4 x^{3} h$ as the second term in the
numerator when computing the deriv of $x^{4}-$ namely, we can get $n-1 x$ 's and one $h$ in $n$ ways, each for each of the $n$ factors $\mathrm{x}+\mathrm{h}$ in $(\mathrm{x}+\mathrm{h})^{n}$.

$$
=\lim _{h \rightarrow 0} \frac{n x^{n-1} h+B x^{n-2} h^{2}+C x^{n-3} h^{3}+\ldots}{h} \quad, \text { since the } \mathrm{x}^{n} \text { cancels }
$$

$=\lim _{h \rightarrow 0} n x^{n-1}+B x^{n-2} h+C x^{n-3} h^{2}+\ldots$, from canceling the h
$=\mathrm{nx}{ }^{n-1}$, no matter what $\mathrm{B}, \mathrm{C}$, etc. are, because as h approaches 0 , all the terms except the first term poof

It is important, and good news, that from now on we have the Power Rule. When we need to diff any power of $x$, we won't have to use the limit formula and go through any calculations, since we already did them, in one fell swoop, when we did the computation directly above in proving the Power Rule.

Two other examples of the Power Rule that we haven't yet seen are:

$$
\begin{aligned}
& \frac{d}{d x} x^{10}=10 x^{9} \\
& \frac{d}{d x} x^{153}=153 x^{152}
\end{aligned}
$$

These can be seen to hold by taking $\mathrm{n}=10$ and $\mathrm{n}=153$, respectively.
Notice how short and simple using the Power Rule is. Here are a few Power Rule pointers, to help remember the Power Rule and to help avoid misconceptions:

1) There's a power in "the answer", but not the same power is in "the problem".
2) But don't forget the coefficient $n$.
3) What happens to $x^{n}$ when it gets diff'd? One way to think of it is: $n$ steps down two ways. First, n gets replaced by n-1. Second, n also demoted from power to coefficient.

Here's another amazing, and fortunate, thing about the Power Rule: It works for noninteger values of $n$, and also for negative values of $n$. We've actually shown it only for positive integers, and we can't yet prove it for, say, $\mathrm{n}=-10$, or for $\mathrm{n}=\sqrt{2}-$ for that we need other formulas and rules which we'll develop later - but it is true for all n .

In some of the exercises from the previous chapter we proved, using the limit definition of deriv, the Power Rule for $\mathrm{n}=-1$ and $\mathrm{n}=\frac{1}{2}$. It's also fun and quick to prove it for $\mathrm{n}=$ 0 . (Hint: You won't need the lim part of the definition of deriv.)

Now let's take a break from calculus, with two review sections - one quickie on fractions, and another on the laws of exponents and how they figure in defining what is meant by non-positive and fractional exponents. After that we'll start in on calculus again.

## Vacation from Calculus (fraction review -- optional)

When I teach calculus I notice that one of big stumbling blocks for students is dealing with fractions. So let's try to nip that stumbling block in the bud with a quick fraction review.

Multiplying and dividing fractions seems to be easier than adding and subtracting them, so let's do multiplying and dividing first. Multiplying is a simple matter of what I call "across the board". To multiply two fractions, multiply the two numerators, and use that as the numerator for the answer. Ditto with denominators. In algebra notation:

$$
\frac{a}{b} \times \frac{c}{d}=\frac{a c}{b d}
$$

It really is "across the board"! In fact, in the formula above, just remove the time-sign in the left side and squoosh the two fractions together, and you'll get "the answer".

For example: $\frac{5}{8} \times \frac{7}{9}=\frac{5 \times 7}{8 \times 9}=\frac{35}{72}$

We could actually prove the above formula. Here's how: Think about what dividing that is, a fraction -- means. Dividing a by b means finding the number x which satisfies $\mathrm{bx}=\mathrm{a}$. That is, the "answer" to a division problem is that number which, when multiplied by the denominator, gives the numerator. So ac divided by bd is that number which, when multiplied by bc, gives ac. And $\frac{a}{b} \times \frac{c}{d}$ fits the bill! Here's how we check that:

$$
\left(\frac{a}{b} \times \frac{b}{d}\right) \times b d=a b, \text { since the } \mathrm{b} \text { and the } \mathrm{d} \text { cancel. }
$$

Now let's deal with division. We want a formula for $\frac{a / b}{c / d}$, or (another way of writing the same thing) $\frac{a}{b} \div \frac{c}{d}$-- that is, we want a "simple" fraction, not a "complex fraction" but a fraction with a single numerator and a single denominator. You probably recall the phrase "invert and multiply". That's a cool way to put it. But what is it that gets inverted? Which fraction, that is? Quick answer: the fraction of the bottom, or the second fraction, c/d. And then, once that fraction is inverted ( meaning, num and denom switched), multiply it by the "other" fraction, $\mathrm{a} / \mathrm{b}$. Here's the formula:

$$
\frac{a / b}{c / d}=a / b \times d / c-\text { which is the invert and multiply rule. }
$$

We can then use the first, "across the board" formula, to get "the answer" $\frac{a d}{b c}$. For example:

$$
\frac{5 / 8}{7 / 9}=5 / 8 \times 9 / 7=\frac{45}{56}
$$

When I was a kid, say fourth or fifth grade, I (already a mathematician!) wanted to convince myself that, say, $5 / 8$ divided by $7 / 9$ actually did mean how many times $7 / 9$ "fit into" $5 / 8$. I drew a "ruler" and worked it out, and indeed it did what I wanted it to do. I won't go into it, but division of fractions actually does make the same kind of sense as division of integers. It's not hard to visualize 8 divided by $2 ; 2$ "gozinta" 84 times. And, yes, $9 / 7$ "gozinta" $5 / 845 / 56$ times. I'm telling you this because I hope that it might help you see that these rules do make sense; they're not arbitrary dictums invented by dictators! The above rule, for example, is not "just one o' those things ya hafta take on faith". You may take it on faith, but you don't have to. When I was a kid mathematician, I didn't.

Back to math (but not calculus!): Next on the program is addition and subtraction of fractions. That's probably easier to understand, as far as the concepts go (the "common denominator business"), but sometimes it takes longer to actually do. For example, let's now try adding our two old friends, $5 / 8$ and $7 / 9$. A common denominator is 72 , so we need to convert (not "reduce", but the opposite) each of $5 / 8$ and $7 / 9$ into some fraction with denominator 721 Here's how it goes:
$5 / 8=? / 72$. Ask, what would we multiply 8 by to get 72 ? Answer: 9 . So that's what we need to multiply the numerator 5 by, to get the new numerator. So $5 \times 9=45$. So $5 / 8=45 / 72$.

Similar stuff with the other fraction, 7/9 (only, different numbers). We still want the new denominator to be 72 . What would we multiply 9 by to get 72? Answer: 8 . So that's what we need to multiply the numerator 7 by , to get the new numerator. So $7 \times 8=56$. So $7 / 9-56 / 72$.

Now that we have both fractions with the same denominator, we can add them. ("Add up the apples", and "the apple" is $1 / 72$.). So "the bottom line", literally, is:

$$
5 / 8+7 / 9=45 / 72+56 / 72=101 / 72
$$

Here's a piece of good news: When searching for a common denominator, it doesn't have to be the lowest common denominator. It can be any common denominator. (However, if we don't use the lowest, the answer is guaranteed to NOT be in lowest terms.)

Back to calculus, sort of. When using the power rule, where the power is a fraction, we'll definitely have to subtract 1 from this fraction. Here's an easy way to do subtract 1 from fraction, and to avoid making careless mistakes: To get the numerator of the power in the answer, subtract the denominator of the "first" fraction from its numerator. And keep the denominator the same. The formula would be:

$$
\mathrm{a} / \mathrm{b}-1=(\mathrm{a}-\mathrm{b}) / \mathrm{b} \text {. }
$$

The formula holds because $b / b$ always equals 1 , no matter what $b$ is. And here are some pointers about what to expect:

1) If the "first power" is a fraction, expect the new power to also be a fraction.
2) If the first power is greater than 1 , expect the new power to be a positive number. For example, try $5 / 4$ as the first power. (The new power will be $1 / 4$.)
3) If the first power is between 0 and 1 , expect the new power to be negative, but not very negative. It should be between -1 and 0 . For example, try $1 / 4$ as the first power. (The new power will be $-3 / 4$.)
4) If the first power is negative, expect the new power to be even more negative. (For example, try $-1 / 4$ as the first power. (The new power will be $-5 / 4$.)

For some problems, it might be helpful to imagine things on "the number line".

## C. Extended Vacation from Calculus (Laws of Exponents -- optional)

You learned these formulas a long time ago, but if you didn't quite catch it, here's another chance. A review can never hurt. Also, if you'd like to know something about why these formulas hold, here's your chance for that. In the following Laws of Exponents (powers), x is any number and m and n are positive integers:

1) $x^{m+n}=x^{m} x^{n}$
2) $x^{m-n}=\frac{x^{m}}{x^{n}}$, if $\mathrm{x} \neq 0$
(3), (4), (6), and (7) are not really formulas; they're definitions. (And we'll show why these definitions make sense.)
3) If $x \neq 0$, then $x^{0}=1$
4) $x^{-n}=\frac{1}{x^{n}}$, if $\mathrm{x} \neq 0$
5) $x^{m n}=\left(x^{m}\right)^{n}$
6) $x^{1 / n}=\sqrt[n]{x}$
7) $x^{m / n}=(\sqrt[n]{x})^{m}=\sqrt[n]{x^{m}}$ (The num "becomes" the power; the denom "becomes" the root.)

These laws hold when $\mathrm{x}, \mathrm{m}$, and n are actual numbers and also when they're variables. Here's why they hold; it boils down to definitions, in particular the definition of powers:

1) Proof that $x^{m+n}=x^{m} x^{n}$ :

$$
\begin{aligned}
\mathrm{x}^{m+n} & =\mathrm{xxx} \ldots(\mathrm{~m}+\mathrm{n}) \mathrm{x} \text { 's, by the definition of powers (Here the juxtaposition } \\
& \text { means that all the x's are multiplied together.) } \\
& =(\mathrm{xxx} \ldots \mathrm{~m} \mathrm{x's})(\mathrm{xxx} \ldots \mathrm{n} \times \mathrm{s}), \text { "splitting up" the } \mathrm{m}+\mathrm{n} \mathrm{x} \text { 's } \\
& =\mathrm{x}^{m} x^{n}, \text { again by the definition of powers }
\end{aligned}
$$

2) Proof that $\mathrm{x}^{m-n}=\frac{x^{m}}{x^{n}}$ :

By (1) directly above, $x^{m}=x^{m-n} x^{n}$, since $m=(m-n)+\mathrm{n}$
Dividing both sides by $\mathrm{x}^{n}$, we get what we want to prove.
3) Motivation for defining $x^{0}=1$ when $x \neq 0$ :

If you wanted to extend the definition of powers to cases where the power is 0 , how would you do it? (You can't say " $x{ }^{0}$ is zero $x$ 's multiplied together. What would that mean?) One of my favorite things in math is extending definitions to numbers which are not positive integers, in ways that make some kind of sense. And in this case, "some kind of sense" means that the first two "laws of exponents" still have to hold - for all n, not only positive integers; we want algebra to be consistent. Below we'll show that there is only one way to do that, for powers that are 0 , negative, or fractional.

So here's why we must define $\mathrm{x}^{0}$ to be 1 ; it basically comes from law (2) - and it works only when $\mathrm{x} \neq 0$ :

$$
\begin{aligned}
x^{0} & =x^{1-1}, \text { simply because } 0=1-1 \\
& =\frac{x^{1}}{x^{1}}, \text { because of law }(2)
\end{aligned}
$$

$=1$, because any number divided by itself is 1
As quick examples, $2^{0}, 7^{0},(-7)$, and $(1 / 7){ }^{0}$ are all 1 .
4) Motivation for defining $x^{-n}=\frac{1}{x^{n}}$

Again, we want law (2) to continue to hold, even for negative powers. So we must have:

$$
\begin{aligned}
x^{-n} & =x^{0-n}, \text { simply because }-\mathrm{n}=0-\mathrm{n} \\
& =\frac{x^{0}}{x^{n}}, \text { because of law (2) } \\
& =\frac{1}{x^{n}}, \text { because of ( } 3 \text { ) directly above }
\end{aligned}
$$

Here are some examples:

$$
2^{-2}=\frac{1}{2^{2}}=\frac{1}{4}
$$

$$
\begin{aligned}
& 2^{-3}=\frac{1}{2^{3}}=\frac{1}{8} \\
& 10^{-3}=\frac{1}{10^{3}}=\frac{1}{1000} \\
& (1 / 6)^{-3}=\frac{1}{(1 / 6)^{3}}=\frac{1}{(1 / 216)}=216
\end{aligned}
$$

Two pointers:
(A) Don't expect the answers to problems involving negative exponents to be negative. As in the third example, the answer might be small but not negative!)
(B) Look at the last example, how large the answer is, "despite" the small base and negative exponent.
(5) Proof that $\left(x^{m)^{n}}=x^{m n}\right.$

As in the proof of (1), the gyst is the definition of exponents.

$$
\begin{aligned}
x^{m n} & =\mathrm{xxx} \ldots \mathrm{mn} \mathrm{x} ' \mathrm{~s}, \text { by the definition of exponents } \\
& =(\mathrm{xxx} \ldots \mathrm{~m} \mathrm{x} s) \ldots \mathrm{n} \text { of these (meaning } \mathrm{n} \text { of the quantity in paratheses) } \\
& =\mathrm{x}^{m} \ldots \mathrm{n} \mathrm{x}^{m} \mathrm{~s}, \text { again by the definition of exponents } \\
& =\left(x^{m}\right)^{n}, \text { one more time by the definition of exponents }
\end{aligned}
$$

(6) Motivation for defining $x^{1 / n}=\sqrt[n]{x}$

In order for our latest law to hold (even though $1 / n$ isn't an integer), we need:

$$
\begin{aligned}
\left(x^{1 / n)^{n}}\right. & =x^{(1 / n) n}, \text { by }(5) \text { directly above } \\
& =x^{1}, \text { simply because }(1 / \mathrm{n}) \mathrm{n}=1 \\
& =\mathrm{x}, \text { because any number to the first power is } 1
\end{aligned}
$$

Now, taking nth roots of both sides, we get the only definition possible of $x^{1 / n}$ :
Namely, $x^{1 / n}=\sqrt[n]{x}$
(7) Motivation for defining, for all fractions $\mathrm{m} / \mathrm{n}, x^{m / n}=(\sqrt[n]{x})^{m}$ :

$$
\begin{aligned}
x^{m / n} & =x^{(1 / n) m}, \text { simply because } \mathrm{m} / \mathrm{n}=(1 / \mathrm{n}) \mathrm{m} \\
& =\left(x^{1 / n}\right)^{m}, \text { in order for }(5) \text { to hold } \\
& =(\sqrt[n]{x})^{m}, \text { by our latest definition (6) }
\end{aligned}
$$

(7') And here's the proof that the second equation in (7) must also hold; that is,
$x^{m / n}=\sqrt[n]{x^{m}}$ We have:

$$
\begin{aligned}
x^{m / n} & =x^{m(1 / n)}, \text { again because } \mathrm{m} / \mathrm{n}=\mathrm{m}(1 / \mathrm{n}) \\
& =\left(x^{m}\right)^{1 / n}, \text { by }(5) \\
& =\sqrt[n]{x^{m}}, \text { by }(6)
\end{aligned}
$$

Note that, although both (7) and (7') are useful, both in mathematical proofs and calculations, it's easier in computations with actual numbers, to use (7), because we wind up dealing with smaller numbrers. For example, using (7) to calculate $8^{2 / 3}$ :

$$
8^{2 / 3}=(\sqrt[3]{8})^{2}=2^{2}=4
$$

whereas the other way - if we used ( $7^{\prime}$ ) - involves a two-figure numbers)

Here's one more example: $16^{3 / 4}=(\sqrt[4]{16})^{3}=2^{3}=8$
And here's an example with a power that's both negative and fractional:

$$
16^{-3 / 4}=\frac{1}{16^{3 / 4}}=\frac{1}{8}, \text { from the problem directly above }
$$

## Getting Serious about Functions, I

In the first section of this book (The Case of the Missing Speedometer), "function" was loosely defined as "a rule which assigns to every number, another number $\mathrm{f}(\mathrm{x})$ ". Necessity was the mother of invention, and we needed the idea of function in order to abstract the concept of speed (which is rate of change of displacement) - that is, in order to define derivative (the limit formula).

This is a good time to get mathematically rigorous about functions. Two of the main reasons for this are: (1) to express formulas and (2) to prove formulas. Up until now we haven't needed to be rigorous about functions because our only formula, the Power rule, is not about functions in general but only about functions of the form $x^{n}$. From now on we will need to stand more on ceremony. We'll need certain notation concerning functions $f(x)$ in general. As you'll see, many of the formulas will have $g$, or $g(x)$, in them, as well as $f(x)$, and sometimes - in order to be consistent with other textbooks only g .

Here's a question: What is the deriv of the function $\mathrm{x}^{2}+x^{3}$ ? Some of you might have done this exercise from Chapter 1, and some of you who haven't might now be tempted to. How would you do that problem? Well, you might be inclined to use intuition and diff, separately, each of $x^{2}$ and $x^{3}$, then add up the two deriv's. That is, since the problem involved a sum, so might the answer. (And indeed it does. The answer is $2 \mathrm{x}+$ $3 x^{2}$.)

Now, what if the problem had been $x^{2}+2 x^{3}$ ? Would you have been, similarly, tempted to answer $2 x+6 x^{2}$ ? And what about $x^{2}-2 x^{3}$ ? If you gave in to temptation, what you did was, unconsciously, use what are known as the Sum Rule of diff-ing, along with the Constant Multiple Rule of diff-ing. You figured, correctly, that the deriv of a sum is the sum of the deriv's, and that the deriv of a constant multiple is that same multiple of the deriv.

So one reason we need to talk about functions in general is to express the Sum Rule and the Constant Multiple Rule. But that's later. For this section, we're dealing only with just-plain functions, no more calculus (promise).

As we've seen, a function is a rule which assigns, to certain numbers, specific values "at" that number (or point) - and that value is a number. (If you think about it, "function", as a noun, means the same thing in math as it does in life. "How you act in a crisis is a function of how you act not in a crisis.") Things that we diff are functions, and so are their deriv's. A colloquial way of defining "function" might be "a rule which tells you what to do with x ".

There are things which every function has, and which are sometimes important to define and identify. Here are two such definitions:
(1) The domain of a function $f(x)$ is defined to be the set of numbers $x$ for which $f(x)$ is defined - that is, it's the set of all possible x's..

Often the domain of a function consists of all of the numbers, from minus-infinity to plus-infinity. But sometimes there are $x$ 's for which $f(x)$ can't be defined. For example, if we take $\mathrm{f}(\mathrm{x})=\frac{1}{x}$, then $\mathrm{f}(\mathrm{x})$ won't "work" for $\mathrm{x}=0$ (because "you can't divide by 0 "), so 0 is not in the domain of "this f ).. Also, if we take (a different function) $\mathrm{f}(\mathrm{x})=\frac{1}{x-1}, \mathrm{f}(\mathrm{x})$ won't "work" for $\mathrm{x}=1$, so 1 is not in the domain of this other f . A different kind of example is the function $\mathrm{f}(\mathrm{x})=\sqrt{x}$; this f 's domain consists of the non-negative numbers, because negative numbers have no (real) square roots.
(2) The range of a function $f(x)$ is defined to be the set of all values that the function can take on - that is, it's the set of all possible $f(x)$ 's.

Again, the range of a function often consists of all numbers, but not always. For example, the function $\mathrm{f}(\mathrm{x})=\frac{1}{x}$, the first example in the last paragraph, doesn't include 0 , because there's no $x$ such that $\frac{1}{x}=0$.

We will now talk about ways of combining two functions, the reason being that notation is important so we can express and prove formulas and theorems. So: given two functions $f(x)$ and $g(x)$, how can we combine them - in order, that is, to get another function? (Warm-up question: How did we combine, in a recent paragraph, $x^{2}$ and $x^{3}$, in order to get $x^{2}+x^{3}$ ? Answer: We added them.) So -

Definition: Given two functions $f$ and $g$, their sum (another function), denoted $f+g$, is the function defined, for any x at which both functions are defined, by:

$$
(\mathrm{f}+\mathrm{g})(\mathrm{x})=\mathrm{f}(\mathrm{x})+\mathrm{g}(\mathrm{x})
$$

In words, the definition says that we define the "sum function" $f+g$ according to the values of each of $f$ and $g$ at the various points $x$. A shorter way to say this is, we add $f$ and g point-wise.

Here's are two examples:
(1) If we take $\mathrm{f}(\mathrm{x})=\mathrm{x}^{2}$ and $\mathrm{g}(\mathrm{x})=x^{3}$, then $(\mathrm{f}+\mathrm{g})(\mathrm{x})=x^{2}+x^{3}$. Simple enough perhaps too simple for our purposes - like, what do we need to stand on ceremony for? But watch this:
(2) If $\mathrm{f}(\mathrm{x})=x^{2}+x$ and $\mathrm{g}(\mathrm{x})=x^{3}-x$, then $\mathrm{f}+\mathrm{g}$ is the function whose values are: $(\mathrm{f}+\mathrm{g})(\mathrm{x})=\left(\mathrm{x}^{2}+x\right)+\left(x^{3}-x\right)=x^{2}+x^{3}$. this "cancellation" helps show that there's something to be gained by considering, once and for all, the sum $\mathrm{f}+\mathrm{g}$, rather than adding up $f(x)$ and $g(x)$ at each point $)$.

Now try to find the sum function $\mathrm{f}+\mathrm{g}$ when $\mathrm{f}(\mathrm{x})=x^{3}+4$ and $\mathrm{g}(\mathrm{x})=2 x^{2}-2$. Next try it with $\mathrm{f}(\mathrm{x})=\mathrm{x}+\sqrt{x}$ and $\mathrm{g}(\mathrm{x})=x^{2}-2 x$. (Don't forget possible "cancellation".)

Now, suppose we have three functions - f, g, and h. How would we define their sum?
Definition: Given three functions $f$, $g$, and $h$, the sum $f+g+h$ is the function whose values are $\mathrm{f}(\mathrm{x})+\mathrm{g}(\mathrm{x})+\mathrm{h}(\mathrm{x})$. (We could also define it this way: $\mathrm{f}+\mathrm{g}+\mathrm{h}=(\mathrm{f}+\mathrm{g})+\mathrm{h}$.)

What if we had any number of functions? Say, n functions? First, how could we denote this? A common notation that mathematicians use, when we don't know how many we're dealing with, is via subscripts - little numbers to the right and under the f (or the x , depending on the context). So we'd call those n functions $f_{1}, f_{2}, f_{3} \ldots f_{n}$

And our definition of their sum would be:
Definition: Given functions $f_{1}, f_{2}, f_{3} \ldots f_{n}$, the sum $f_{1}+f_{2}+f_{3} \ldots+f_{n}$ is defined to be the function: $\left(f_{1}+f_{2}+f_{3} \ldots+f_{n}\right)(\mathrm{x})=f_{1}(x)+f_{2}(x)+f_{3}(x) \ldots f_{n}(x)$, for all points x at which all the functions are defined.

Example: Suppose each of the $\mathrm{f}_{i}(\mathrm{x})=\mathrm{x}$ (just-plain x ) for all I between 1 and n . What would the sum be? (Answer: nx)

What else can we do with two functions? Here's a quick answer: We can subtract them, multiply them, take any linear combination of them, including multiply just one of them by a constant, divide them (at points where the second function does not have the value zero). So we can talk about $\mathrm{f}-\mathrm{g}, \mathrm{fg}, 3 \mathrm{f}-7 \mathrm{~g}$, and $\mathrm{f} / \mathrm{g}$. Here's what all that looks like when we stand on ceremony:

Definition of Constant Multiple of a Function: Given a function f and a number C, the function Cf is defined by: $(\mathrm{Cf})(\mathrm{x})=\mathrm{Cf}(\mathrm{x})$, for all x at which f is defined.

Definition of the Difference of Two Functions: Given functions $f$ and $g$, the difference $f-g$ is defined by: $(\mathrm{f}-\mathrm{g})(\mathrm{x})=\mathrm{f}(\mathrm{x})-\mathrm{g}(\mathrm{x})$, for all x at which both f and g are defined.

Definition of the Product of Two Functions: Given functions $f$ and $g$, the product $f g$ is defined by: $(f g)(x)=f(x) g(x)$, for all $x$ at which both $f$ and $g$ are defined.

Definition of the Square of a Function: Given a function $f$, the square $f^{2}$ of $f$ is defined by: $\left(\mathrm{f}^{2}\right)(\mathrm{x})=[\mathrm{f}(\mathrm{x})]^{2}$ (Maybe too many parentheses for comfort, but the key, again, is "point-wise".)

Similarly with the cube of a function, fourth power, or any power - or any root.

Definition of the Quotient of Two Functions: Given functions $f$ and $g$, the quotient $f / g$ is defined by: $(f / g)=f(x) / g(x)$, for all $x$ at which both $f$ and $g$ are defined, and at which $\mathrm{g}(\mathrm{x}) \neq 0$.

AND we can combine the combinations! For example, we can first multiply, then add, then add those two - to get $\mathrm{fg}+\mathrm{f}+\mathrm{g}$. We can consider $2 \mathrm{f} / \mathrm{g}$, or $2 \mathrm{~g} / \mathrm{f}$, or $(\mathrm{f}+\mathrm{g}) /(\mathrm{f}-\mathrm{g})$. There's no end! We'd keep getting new functions (maybe some of them would happen to be the same, though).

At this point we can say "anything we can do with numbers, we can do with functions". We might also add "and then some". For there's one thing that's strictly for functions. That thing is the subject of the next section.

## Getting Serious about Functions, II

What else can we possibly do (without invoking calculus)? Well, we can follow one by the other. That is, given two functions f and g , watch what can happen: g (being a function) acts on any number $x$ (in its domain), giving us a number $g(x)$ - and then $f$ can get into the act, by (literally) acting on $\mathrm{g}(\mathrm{x})$ (when $\mathrm{g}(\mathrm{x})$ is in its domain).

Now let's stand on ceremony:
Definition of the composition $\mathrm{f}(\mathrm{g})$ : Given two functions f and g (in that order), the composition (function) $f(g)$ is defined to be the function: $[f(g)](x)=f[g(x)]$, for any $x$ which is in the domain of $g$, and for which $g(x)$ is in the domain of $f$.

This is newer than the functional operations in the previous sections, so let's be extrafriendly! First, it might help to think of $f$ and $g$ as verbs, and to say "first we $g$, then we $f$ ". Or "first we run $x$ through the $g$-machine, then we run the resulting $g(x)$ through the $f$ machine". One machine at a time. The whole composition $\mathrm{f}(\mathrm{g})$ is a "wha' happens next machine", sometimes know as a Rube Goldberg machine, after the cartoonist who drew cartoons about them.

Here are a few examples:
(1) $f(x)=3 x, g(x)=2 x$

$$
\begin{aligned}
\text { Then }[\mathrm{f}(\mathrm{~g})](\mathrm{x}) & =\mathrm{f}[\mathrm{~g}(\mathrm{x})]=\mathrm{f}(2 \mathrm{x}) \text {, because first we } \mathrm{g} \\
& =3(2 \mathrm{x}) \text {, because then we } \mathrm{f} \text {, but remember, what we } \mathrm{f} \text { is } 2 \mathrm{x} \text { (not } \mathrm{x} \text { ) } \\
& =\underline{6 x}, \text { by just-plain algebra }
\end{aligned}
$$

(2) $f(x)=x+1, g(x)=2 x$

$$
\text { Then } \begin{aligned}
{[\mathrm{f}(\mathrm{~g})](\mathrm{x}) } & =\mathrm{f}[\mathrm{~g}(\mathrm{x})]=\mathrm{f}(2 \mathrm{x}) \\
& =\underline{2 \mathrm{x}} \pm \underline{1}-\text { again, remember, what we } \mathrm{f} \text { is } 2 \mathrm{x}
\end{aligned}
$$

(3) (switching the f with the g ) $f(x)=2 x, g(x)=x+1$

$$
\text { Then } \begin{aligned}
{[\mathrm{f}(\mathrm{~g})])(\mathrm{x}) } & =\mathrm{f}[\mathrm{~g}(\mathrm{x})]=\mathrm{f}(\mathrm{x}+1) \\
& =2(\mathrm{x}+1)-\text { again, what we } \mathrm{f} \text { is } \mathrm{x}+1 \\
& =\underline{2 x} \pm \underline{2}
\end{aligned}
$$

Note that the function we use first is the one closest to the x in the notation $[\mathrm{f}(\mathrm{g})](\mathrm{x})$
Before we do more examples, look again at both (2) and (3). How do the problems differ? (Answer: switcheroo) And do the answers differ? (yes) In general, $f$ of $g$ is not the same as $g$ of $f$. (Sometimes it is. (1) provides an example of this. But in general, it matters,
which order the functions are in. Composition is NOT commutative, like addition and multiplication.) Now for one more example, to get squaring into the deal:
(4) $f(x)=x^{2}, g(x)=2 x$

$$
\begin{gathered}
{[\mathrm{f}(\mathrm{~g})](\mathrm{x})=\mathrm{f}[\mathrm{~g}(\mathrm{x})]=\mathrm{f}(2 \mathrm{x})} \\
=(2 \mathrm{x})^{2} \\
=4 \mathrm{x}^{2}
\end{gathered}
$$

Composition leads to one more functional operation -- on one function (which we'll call f). Some functions (not all) have what we'll call inverse functions, denoted $\mathrm{f}^{-1}$. (Yes, this notation is the same as for the power of -1 - every once in a while, mathematical notation gets, unfortunately, duplicated and mathematicians, in this case, have not resolved that quandary. In order to tell what operation we mean - power -1 or inverse - we have to consider the context. In this book we won't have much trouble; the context will be clear, and I'll mention it if it's not.)

So, inverse: Given a function $f$, we can ask, is there a function $g$ such that for every x in the domain of $\mathrm{f},[\mathrm{g}(\mathrm{f})](\mathrm{x})=\mathrm{x}$ ? That is, is there a g which un-does f ? Another way to put all this is to first define something called the identity function - call it $\mathrm{I}-\mathrm{by} \mathrm{I}(\mathrm{x})=\mathrm{x}$, for all $x$. (In the Cartesian plane, that function's graph is the main diagonal - with points such as $(1,1),(2,2),(-100,-100)$ and, yes, the origin.) So asking about an inverse of a function $f$ is asking, is there a function $g$ such that $g(f)=I$ ?

Here, to conclude this section, are some examples of inverses of functions, and a couple of theorems:
(1) The inverse of $f(x)=x+2$ is $g(x)=x-2$. We know this because, for all $x$, we have $g[f(x)]=g(x-2)=(x-2)+2=x$, so $f$ undoes $g$.
(2) The inverse of $f(x)=3 x$ is $g(x) / x / 3$. We know this because, for all $x$, we have $g[f(x)]=g(3 x)=3 x / 3=x$.

How do we actually find inverses? Well, often intuition does it - it depends on the problem, and on the individual student. If intuition doesn't work, here's a way to do it, step by step:

Step A:
Since we're given $f(x)$, we can set up the equation: $f(y)=x$. (Note where the $y$ appears in that equation.)

Step B:

Now that we have an equation, we can solve it for y . (Yes, y . Also, we're solving for y in terms of $x$ - we don't have to know x as a number.)

Step C: Now you have another equation, $\mathrm{y}=$ some function of x . That function is the inverse of $f$-- our answer!

Here's an example of using that method: Find the inverse function of $f(x)=2 x+3$.
(A) $f(y)=x$, so we have $2 y+3=x$.
(B) Solving for y , we get:

$$
\begin{aligned}
2 y & =x-3 \\
y & =(x-3) / 2
\end{aligned}
$$

(C) We now have our answer:

$$
\begin{aligned}
g(x) & =f^{-1}(x) \\
& =(x-3) / 2
\end{aligned}
$$

Another more visual way to think of inverses of functions is to first imagine the graph of the function in the Cartesian plane. Think first of the "vertical line rule" about the functions: a set of points (usually a line, straight or not) in the $x-y$ plane is the graph of a function if and only if every vertical line meets that set in one and only one point. Then we can say that a function has an inverse if and only if, besides obeying that "vertical line rule", the graph of that function also obeys the "horizontal line rule", meaning every horizontal line meets the graph in one and only one point).

We won't be using this concept of inverse for awhile and we won't be using it very often. But it's important. We'll need it to diff functions like $2^{x}$ (which we haven't done yet), and to talk about and diff "inverse trig" functions. In general, in order to maximize our "function vocabulary", we'll need to not neglect functions which arise from other functions, and inverses are examples of these.

## The Return of Calculus, I

Now that we're familiar with the idea of functions, what can be done with them, and the notation involved, we're ready to use what we know to express - and thereby more easily
understand, remember, and visualize - all the various diff-ing rules. This section will start us off on this project.

Two sections back we already talking about diff-ing functions like $2 \mathrm{x}^{2}+3 x^{3}$. We talked about how, intuitively and perhaps unconsciously, if given that problem, we'd first diff each of $x^{2}$ and $x^{3}$, and then combine those two deriv's in the same way that "the original" $x^{2}$ and $x^{3}$ were combined in the problem. So we'd get $2(2 x)+3\left(3 x^{2}\right)=4 x+$ $9 x^{2}$ as our answer.

This, again, was done out of intuition. If we'd had another problem to do - for example, $2 x^{7}+3 x^{9}-$ we might have done something similar -- that is, first diff each of $x^{7}$ and $x^{9}$, then combine the two deriv's using the same "principle" that we used in diff-ing $2 x^{2}+3 x^{3}$.

And now let's ask ourselves, what is that principle? The answer to that question appears directly below, in the form of four theorems - the first three are identical to those which appear in most calc texts, and the last embodies those three. (If you understand and feel comfortable with that last, you can use it and forget about the other three):

THE SUM RULE: For any two functions $f$ and $g,(f+g)^{\prime}=f^{\prime}+g$ ' (I call this "divide and conquer").

THE DIFFERENCE RULE: For any two functions $f$ and $g,(f-g)^{\prime}=f^{\prime}-g^{\prime}$ (This is also divide and conquer.)

THE CONSTANT MULTIPLE RULE: For any function f and any number C ,
$(\mathrm{Cf})^{\prime}=\mathrm{Cf}^{\prime}$ (Another divide and conquer, or maybe just whittle down - Note that C has to be a number/constant, not a non-constant function.)

THE LINEAR COMBO RULE: For any two functions f and g , and any two numbers C and $\mathrm{D},(\mathrm{Cf}+\mathrm{Dg})^{\prime}=\mathrm{Cf}^{\prime}+\mathrm{Dg}{ }^{\prime}$ (Divide and conquer again -- Note that C and D have be numbers, not non-constant functions.)

The two examples given at the beginning of this section are examples of using the Linear Combo Rule. Here are some more examples:
(A) Example of using the Sum Rule:

$$
\left(\mathrm{x}^{100}+x^{101}\right)^{\prime}=\left(x^{100}\right)^{\prime}+\left(x^{101}\right)^{\prime}=100 \mathrm{x}^{99}+101 x^{100}
$$

(B) Example of using the Difference Rule:

$$
\left(\mathrm{x}^{100}-x^{101}\right)^{\prime}=\left(x^{100}\right)^{\prime}-\left(x^{101}\right)^{\prime}=100 \mathrm{x}^{99}-101 x^{100}
$$

(C) Example of using the Constant Multiple Rule:

$$
\left(7 x^{100}\right)^{\prime}=7\left(x^{100}\right)^{\prime}=7\left(100 x^{99}\right)=700 x^{99}
$$

(D) Example of using the Linear Combo Rule:

$$
\begin{aligned}
& \left(7 \mathrm{x}^{100}+3 x^{101}\right)^{\prime}=7\left(\mathrm{x}^{100}\right)^{\prime}+3\left(\mathrm{x}^{101}\right)^{\prime}=7\left(100 \mathrm{x}^{99}\right)+3\left(101 \mathrm{x}^{100}\right)= \\
& 700 \mathrm{x}^{99}+303 \mathrm{x}^{100}
\end{aligned}
$$

A tip about using the Constant Multiple Rule (and therefore the Linear Combo Rule) when it applies to constant multiples of powers of x : After awhile you'll find yourself automatically just multiplying the exponent by the coefficient, then using that as the coefficient for whatever power of x .

Possibly you can see intuitively why these rules hold, and their short mathematical proofs reflect this intuition. Both proofs rely on definitions - of deriv, of sums of functions, of constant multiples of functions, and so on:

Proof of the Sum Rule:

$$
\begin{aligned}
&(\mathrm{f}+\mathrm{g})^{\prime}(\mathrm{x})=\lim _{h \rightarrow 0} \frac{(f+g)(x+h)-(f+g)(x)}{h}, \text { by "the def of diff" } \\
&=\lim _{h \rightarrow 0} \frac{[(f)(x+h)+g(x+h)]-[f(x)+g(x)]}{h}, \text { by the definition of the sum } \mathrm{f}+\mathrm{g}, \\
& \text { acting on both } \mathrm{x}+\mathrm{h} \text { and on just-plain }
\end{aligned}
$$

x

$$
=\lim _{h \rightarrow 0} \frac{[(f)(x+h)-f(x)]+[g(x+h)-g(x)]}{h \quad} \text {, by re-arranging the terms in the }
$$

$$
\begin{gathered}
=\lim _{h \rightarrow 0} \frac{(f)(x+h)-f(x)]}{h}+\frac{g(x+h)-g(x)}{h}, \\
\text {, by simple algebra, again to get } \\
\text { where we're headed }
\end{gathered}
$$

$$
=\lim _{h \rightarrow 0} \frac{(f)(x+h)-f(x)]}{h}+\lim _{h \rightarrow 0} \frac{g(x+h)-g(x)}{h} \text {, because the lim of a sum is }
$$

$=f^{\prime}(x)+g^{\prime}(x)$, by "the def of diff"
$=(\mathrm{f}+\mathrm{g})^{\prime}(\mathrm{x})$, by the definition of the sum of two functions
Since this is true for all $x$, we have (from the definition of functions in general):

$$
(\mathrm{f}+\mathrm{g})^{\prime}=\mathrm{f}^{\prime}+\mathrm{g}^{\prime}, \text { and our proof is finished. }
$$

Proof of the Constant Multiple Rule:

$$
\begin{aligned}
(\mathrm{Cf})^{\prime}(\mathrm{x})= & \lim _{h \rightarrow 0} \frac{(C f)(x+h)-(C f)(x)}{h}, \text { by "the def of diff" } \\
& =\lim _{h \rightarrow 0} \frac{C f(x+h)-C f(x)}{h}, \text { by the definition of constant multiple of a function } \\
& =\lim _{h \rightarrow 0} C\left[\frac{f(x+h)-f(x)}{h}\right], \text { by algebra (factoring out the C) } \\
= & C \lim _{h \rightarrow 0} \frac{f(x+h)-f(x)}{h}, \begin{array}{l}
\text { because the lim of a constant multiple is the same as } \\
\text { the constant multiple of the lim (more about that in } \\
\text { the later section on "getting serious about limits") }
\end{array}
\end{aligned}
$$

$=\mathrm{C}\left[\mathrm{f}^{\prime}(\mathrm{x})\right]$, by "the def of diff"
$=\left(\mathrm{Cf}^{\prime}\right)(\mathrm{x})$, by the definition of constant multiple of a function
Again, since this is true for all x , we have, from the definition of function, $(\mathrm{Cf})^{\prime}=\mathrm{Cf}^{\prime}$, and both our proofs are complete.

Note again that the Constant Multiple Rule is true only when C is a constant. ( C can't be a function other than a constant.) In the next chapter we'll go over what to do when we have the product of two functions. For now - the next section will deal with something
called The General Power Rule. That's when, instead of just-plain $x$ being taken to some power, and then diff'd, we take any function of $x$.

## More Powers That Be (The General Power Rule)

What can we diff so far? Still only powers, and linear combo's of powers. But there are many more functions to find deriv's of; in particular, there are other ways of combining powers besides linearly. Like, what if we take some power of a polynomial? Examples of this are: $\left(x^{2}+1\right)^{9},\left(3 x^{4}-6 x^{3}-12 x\right)^{2 / 3},\left(10 x^{100}+4 x\right)^{-5}$

All of these functions are of the form, $[\mathrm{g}(\mathrm{x})]^{n}$, where $\mathrm{g}(\mathrm{x})$ is some function of x (in our examples above, a polynomial), and n is a number, and not necessarily a positive number; it could be, as in two of the examples above, fractional or negative. $g(x)$ is the notation used in most texts, and I like to say, "g for general". g is also for "giant leap", or "glob", meaning a whole big glob. (It could be even bigger than in the examples above.)

For short, it's sometimes helpful to write $g$ rather than $g(x)$. But we will need to remember that g is a function. Anyway, how can we diff $\mathrm{g}^{n}$ ? Is there some kind of rule, for finding ( $\mathrm{g}^{n}$ ), when we know $\mathrm{g}^{\prime}$ ? Something like the Sum and Constant Multiple Rules?

The answer is yes. Since diff-ing means finding rate of change, it might help to ask, how is $g^{n}$ changing? Well, first, as $x$ changes, $g$ (without the power $n$ ) changes, and that change is expressed, as we know, by g '. But second, our "whole function" g " is also changing, according to how $g$ changes. This second change of $g^{n}$ with respect to $g$ (not x ) is, as we know from the Power Rule, equal to $\mathrm{ng}^{n-1}$, and is not the same thing as the change in g with respect to $x$.

So perhaps it's not very surprising that the change of the function $\mathrm{g}^{n}$ can be calculated according to:

$$
\left(\mathrm{g}^{n}\right)^{\prime}=\frac{d g^{n}}{d g} \times \frac{d g}{d x}=\mathrm{ng}^{n-1} \mathrm{~g}^{\prime} \quad-- \text { which gives us - }
$$

THE GENERAL POWER RULE: If $\mathrm{g}=\mathrm{g}(\mathrm{x})$ is any function of x , and if n is any number, then we have: $\left(\mathrm{g}^{n}\right)^{\prime}=\mathrm{ng}^{n-1} \mathrm{~g}^{\prime}$ - at points x where $\mathrm{g}^{\prime}(\mathrm{x})$ exists.

The mathematically rigorous proof of this will be omitted for now, because it's a special case of an even more general diff-ing rule called The Chain Rule, which will be proven later. (I sometimes call it the General General Rule because it's general in two ways - it tells how to diff a composition $\mathrm{f}(\mathrm{g})$, where f could be a function other than a power, and where $g$ could likewise by any function. Also, because it tells how to diff a composition, I also sometimes call it The Composition Rule. But The Chain Rule is a good, descriptive, term for it, and the one which most texts and mathematicians use.)

Here's how to diff the examples given in the beginning of this section, using the General Power Rule:
(1) $\frac{d}{d x}\left(\mathrm{x}^{2}+1\right)^{9}=9\left(x^{2}+1\right)^{8} 2 x$, because we can take $\mathrm{g}(\mathrm{x})$ to be $\mathrm{x}^{2}+1$ and n to be 9
$=18 \mathrm{x}\left(\mathrm{x}^{2}+1\right)^{8}$, by algebra simplification (It's usually nice to put what I call "the little guys" like $x$ to the left, so they don't get lost or forgotten, especially if they'll need to be used for further calculation.)
(2) $\frac{d}{d x}\left(3 x^{4}-6 x^{3}-12 x\right)^{2 / 3}=\frac{2}{3}\left(3 x^{4}-6 x^{3}-12 x\right)^{-1 / 3}\left(12 x-18 x^{2}-12\right)$
(3) $\frac{d}{d x}\left(10 x^{100}+4 x\right)^{-5}=-5\left(10 x^{100}+4 x\right)^{-6}\left(1000 x^{99}+4\right)$

Here are some General Power Rule pointers. Some of them are like the (ordinary) Power Rule pointers:
(1) I sometimes like to say "copy" for $g$, because it gives verbal instructions on how to diff functions of the form $\mathrm{g}^{n}$, when g is a function that we already know how to diff. I say, " n times copy to the $\mathrm{n}-1$ times copy-prime". ) What are we copying? Answer: g)
(2) Or we could say glob instead of copy.
(3) What the General Power Rule essentially says is: First pretend you think $g$ is just "like x " (some independent variable, rather than a function) - that would give us the incorrect $\mathrm{ng}^{n-1}$, BUT THEN throw in $\mathrm{g}^{\prime}$, to give us the correct $\mathrm{ng}^{n-1} \mathrm{~g}^{\prime}$.
(4) And don't forget that exponent n-1 !!!!!!!!!
(5) As in the ordinary Power Rule, "n steps down two ways".
(6) Only one thing happens at a time, or rather in each factor of the answer. Does that make sense?! Well, g either gets diff'd or raised to the power n-1. Never both. Things might have gotten more complicated since the ordinary Power Rule, but again, only one thing at a time.

## That Big Plus (The Product Rule)

This is the rule that tells how to diff the product of two functions (even if both are not constant functions), when we already know how to diff each of the two. How, for example, would we diff the product:

$$
\left(3 x^{5}-2 x^{2}+4\right)^{10}\left(4 x^{4}+3 x^{2}\right)^{17}
$$

of $\left(3 x^{5}-2 x^{2}+4\right)^{10}$ and $\left(4 x^{4}+3 x^{2}\right)^{17}$ ? That product function doesn't fall into any of the categories covered so far. Unless we actually FOIL - ten $\left(3 x^{5}-2 x^{2}+4\right)$ 's and seventeen ( $4 \mathrm{x}^{4}+3 x^{2}$ )'s, something basically completely unfathomable and not even remotely in our plan - this product of powers is, for all practical purposes, neither a polynomial nor a power of a polynomial. It certainly isn't a single power of x. So we need some other trick - a new rule. Here it is:

PRODUCT RULE: Given any two diff-able functions f and g , we have $(\mathrm{fg})^{\prime}=\mathrm{f}^{\prime} \mathrm{g}+\mathrm{fg}{ }^{\prime}$, at all points where both $f$ and $g$ are diff-able.

When I teach the Product Rule in my Calculus classes, I begin the day by writing the following on the board - and it covers the entire board; I make it as big as possible:
1
1
1
1
1
1
1
1
1
1

11111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111
"What's this?" I ask the class.
Students make all sorts of fun guesses (like "the x-y axes", or "the Red Cross), but no, it's "that big plus" - part of the Product Rule, and an important part! I then write the Product Rule formula, incorporating the big plus I just drew. Here's what the board now looks like:


I then own up; I erase and write the Product Rule in the grown-up style of the previous page: $(f g)^{\prime}=f^{\prime} g+f^{\prime}$ - reminding the class that this plus is very important; the Product Rule involves calculating two terms, f'g and fg' (not only the first one). So consider yourself warned!

Before showing why this rule holds, let's see how it's used to diff the function above:
In the Product Rule, take:
$\mathrm{f}(\mathrm{x})=\left(3 \mathrm{x}^{5}-2 x^{2}+4\right)^{10} \quad$ and $\quad \mathrm{g}(\mathrm{x})=\left(4 x^{4}+3 x^{2}\right)^{17}$
Also in the Product Rule, we'll need both $f^{\prime}$ and $g^{\prime}$, and we get them via the General Power Rule:

$$
\mathrm{f}^{\prime}(\mathrm{x})=10\left(3 \mathrm{x}^{5}-2 x^{2}+4\right)^{9}\left(15 x^{4}-4 x\right) \quad \mathrm{g}^{\prime}(\mathrm{x})=17\left(4 \mathrm{x}^{4}+3 x^{2}\right)^{16}\left(16 x^{3}+6 x\right)
$$

We now have four quantities $-\mathrm{f}, \mathrm{g}, \mathrm{f}^{\prime}$, and g ' - as on the right side of the Product Rule. Next we combine them according to the right side of the Product Rule. One way to think of this is: Look at the four quantities as laid out below (ignoring all the words). Then do what I call "cross-multiplying". Indeed we will be multiplying, twice, each function by the one diagonally across from it. Whether or not you choose to think of things that way, our answer - before algebraic simplification - is:

$$
\begin{aligned}
\mathrm{f}^{\prime} \mathrm{g}+\mathrm{fg} & = \\
& 10\left(3 \mathrm{x}^{5}-2 x^{2}+4\right)^{9}\left(15 x^{4}-4 x\right)\left(4 x^{4}+3 x^{2}\right)^{17}+ \\
& +\left(3 \mathrm{x}^{5}-2 x^{2}+4\right)^{10} 17\left(4 \mathrm{x}^{4}+3 x^{2}\right)^{16}\left(16 x^{3}+6 x\right)
\end{aligned}
$$

Simplifying is brief; it's only a matter of "rescuing" the 17 in the second term from the middle of the muddle, and bringing it to the front (so it doesn't get "lost"). Our final answer is:
$10\left(3 \mathrm{x}^{5}-2 x^{2}+4\right)^{9}\left(15 x^{4}-4 x\right)\left(4 x^{4}+3 x^{2}\right)^{17}+$ $+17\left(3 \mathrm{x}^{5}-2 x^{2}+4\right)^{10}\left(4 \mathrm{x}^{4}+3 x^{2}\right)^{16}\left(16 x^{3}+6 x\right)$

Long, I know! But most of the work was in the length, meaning copying down, not so much in calculating.

Note: We could play with it a little more, algebraically (not calculus-ly), as some texts emphasize. What could we do? Well, if you stare for awhile at the above long expression, You might notice some repetition. The $3 \mathrm{x}^{5}-2 x^{2}+4$ and the $4 x^{4}+3 x^{2}$ appear in both terms (raised to different powers). That means we can factor! More specifically, we can factor out the smaller powers. Here's what would result, another way of expressing the answer to our original problem of diff-ing the product of the two functions:

$$
\left(3 x^{5}-2 x^{2}+4\right)^{9}\left(4 x^{4}+3 x^{2}\right)^{16}\left\{10\left(15 x^{4}-4 x\right)\left(4 x^{4}+3 x^{2}\right)+17\left(3 x^{5}-2 x^{2}+4\right)\left(16 x^{3}+6 x\right)\right\}
$$

Well, it is shorter. Later on in this book -promise - we'll see some easier, and in some ways more interesting, examples of the Product Rule. That will happen when we get deriv's of more functions.

Throughout this book we'll see lots of instances where, not only calc, but also algebra comes into play. Thus calc is a good opportunity to review algebra, although sometimes it can feel overwhelming. It's usually okay to concentrate on the calc. Here's an example where there's opportunity for slightly more algebra (and arithmetic). The problem is to diff the function: $\left(3 x^{5}-2 x^{2}+4\right)^{10}\left(4 x^{4}+3\right)^{17}$. (It's almost the same as the problem we just did; the slight difference is that the $x^{2}$ at the very end has been omitted.) As before, the set-up might be:
$f(x)=\left(3 x^{5}-2 x^{2}+4\right)^{10}$

$$
g(x)=\left(4 x^{4}+3\right)^{17}
$$

$$
\mathrm{f}^{\prime}(\mathrm{x})=10\left(3 x^{5}-2 x^{2}+4\right)^{9}\left(15 x^{4}-4 x\right)
$$

$g^{\prime}(x)=17\left(4 x^{4}+3\right)^{16}\left(16 x^{3}\right)$

The essential difference is the $16 \mathrm{x}^{3}$ in $\mathrm{g}^{\prime}$, how small it is. We could, in fact, right away simplify $g$ ' to $272 x^{3}\left(4 x^{4}+3\right)^{16}$. By using the "cross-multiplying" method as in the previous problem, we get:

$$
10\left(3 x^{5}-2 x^{2}+4\right)^{9}\left(15 x^{4}-4 x\right)\left(4 x^{4}+3\right)^{17}+\left(3 x^{5}-2 x^{2}+4\right)^{10}\left(272 x^{3}\right)\left(4 x^{4}+3\right)^{16}
$$

Again, in the second term, we could simplify still further by "rescuing" the little guys 272 and $x^{3}-$ from the middle of the muddle, and placing them at the front of that middle term. We'd get a better expression of the answer:
$10\left(3 x^{5}-2 x^{2}+4\right)^{9}\left(15 x^{4}-4 x\right)\left(4 x^{4}+3\right)^{17}+272 x^{3}\left(3 x^{5}-2 x^{2}+4\right)^{10}\left(4 x^{4}+3\right)^{16}$
Note: There are two kinds of "little guys" - numbers (like 272) and powers of x (like $x^{3}$ ). Move them all to the front.

Here are some Product Rule pointers:

1) It's not like the Sum Rule. (The deriv of a product is in general NOT the same as the product of the deriv's.)
2) Don't forget "that big plus" There are always two terms in the answer to a Product Rule problem.
3) The Product Rule is symmetric in $f$ and $g$, since fg is symmetric in $f$ and $g$. (So justplain f'g can't be the deriv of fg.)
4) The Product Rule doesn't work like the Constant Multiple Rule. The right side of the latter involves only one term. (The Constant Multiple Rule could be thought of as a special case of the Product rule, where $f$ is a constant function.)
5) In "the answer" of a Product Rule problem, each of $f$ and $g$ takes its turn/term getting diff'd.
6) The Product Rule involves three products (one in the problem, two in the answer).
7) Sometimes the two terms in the answer seem unbalanced, one longer than the other. One way this can happen is when $f$ or $g$ is a "little guy". For example, if $f(x)=x$ and $g(x)$ $=\left(3 x^{4}-5 x^{2}\right)^{7}$, then we get:

$$
\frac{d}{d x} x\left(3 x^{4}-5 x^{2}\right)^{7}=\left(3 x^{4}-5 x^{2}\right)^{7}+7 x\left(3 x^{4}-5 x^{2}\right)^{6}\left(12 x^{3}-10 x\right)
$$

since the deriv of x is a mere 1 . Thus the answer is a bit "unbalanced", but correct. So don't panic when that happens.
8) Don't (unless you're a masochist) use the Product Rule when the Constant Multiple Rule will do the trick.
9) If you've previously seen the Product Rule written as (fg)' $=\mathrm{fg}{ }^{\prime}+\mathrm{f}^{\prime} \mathrm{g}$, or as
$(\mathrm{fg})^{\prime}=\mathrm{fg}^{\prime}+\mathrm{gf}{ }^{\prime}$, or any of the other ways of switching around the summands and factors, be assured that all the ( 8 total) ways of writing it are equivalent (and correct). That's because both addition and multiplication are both commutative. The Product Rule as it appears here has advantages which will show up in some of the Exercises at the end of this chapter, and in the next "Quotient Rule section"

Are you curious as to why the Product Rule works? If you are, here's the short answer:
When we ask for the deriv of fg , what we're asking is: How does fg change? Well, f and $g$ both change, and we could think of the scenario like this: First $f$ changes while $g$ stays constant, and then (the other way) g changes while f stays constant. So we can "use" the Constant Multiple Rule twice. What happens first: g stays constant so that part of the deriv of fg becomes f'g. What happens next: f stays constant so that part of the deriv of fg becomes fg'. Add up the two changes and we get the Product Rule.

To conclude this section: If you'd like a more mathematically rigorous proof of the Product Rule, here it is. Like our other recent proofs, it uses definitions - of deriv and of limit - and a cute little algebra trick:
$(f g)^{\prime}=\lim _{h \rightarrow 0} \frac{(f g)(x+h)-(f g)(x)}{h}$, by the definition of the deriv of fg

$$
\begin{aligned}
& =\lim _{h \rightarrow 0} \frac{f(x+h) g(x+h)-f(x) g(x)}{h}, \text { by the definition of the function } \mathrm{fg} \\
& =\lim _{h \rightarrow 0} \frac{f(x+h) g(x+h)-f(x) g(x+h)+f(x) g(x+h)-f(x) g(x)}{h}
\end{aligned}
$$

by adding and subtracting the same quantity - that's the algebra trick I mentioned

$$
=\lim _{h \rightarrow 0} \frac{f(x+h) g(x+h)-f(x) g(x+h)}{h}+\lim _{h \rightarrow 0} \frac{f(x) g(x+h)-f(x) g(x)}{h},
$$

because the lim of a sum is the sum of the lim's
$=\lim _{h \rightarrow 0} g(x+h) \lim _{h \rightarrow 0} \frac{f(x+h)-f(x)}{h}+f(x) \lim _{h \rightarrow 0} \frac{g(x+h)-g(x)}{h}$,
by factoring out $\mathrm{g}(\mathrm{x}+\mathrm{h})$ in the first term, and $\mathrm{f}(\mathrm{x})$ in the second, and because the lim of a product is the product of the lims
$=\mathrm{g}(\mathrm{x}) \mathrm{f}^{\prime}(\mathrm{x})+\mathrm{f}(\mathrm{x}) \mathrm{g}^{\prime}(\mathrm{x})$, by the def of diff, and because $\lim _{h \rightarrow 0} g(x+h)=g(x)$
$=\left(f^{\prime} g+f g g^{\prime}\right)(x)$, by the definition of the product functions $f^{\prime} g$ and $f g{ }^{\prime}$, and also by the definition of the sum of two functions

## That Big Minus (The Quotient Rule)

The previous section, on The Product Rule, dealt with multiplication. This section - on The Quotient Rule - deals with division, or quotients. This last diff-ing rule of this chapter tells how to diff the quotient of two functions when we already know how to diff each of the functions. For example, how would we diff the quotient function,
$\left(3 x^{5}-2 x^{2}\right)^{10} /\left(4 x^{4}+3 x^{2}\right)^{17}$
We already know how to diff the numerator and the denominator, by using the General Power Rule. Here's the rule that tells how to diff the "whole quotient function":

The Quotient Rule: Given any two functions fand $g$ : $\quad(f / g)^{\prime}=\left(f^{\prime} g-g^{\prime}\right) / g^{2}$
at any point x where both $\mathrm{f}^{\prime}$ and $\mathrm{g}^{\prime}$ are defined, and where $\mathrm{g}(\mathrm{x}) \neq 0$.
First note that "the answer" in the Quotient Rule differs from that in the Product Rule in just two ways: (1) The "big plus" is replaced by a "big minus". (2) There's something in the denominator - namely, $\mathrm{g}^{2}$.

Before seeing why this rule holds, let's do an example - namely, the problem posed at the very beginning of this section. First, we can lay it out the way the multiplication problems were laid out in the previous section:

Set $\mathrm{f}=\left(3 x^{5}-2 x^{2}\right)^{10}$

$$
\mathrm{g}=\left(4 x^{4}+3 x^{2}\right)^{17}
$$

Then $\mathrm{f}^{\prime}=10\left(3 x^{5}-2 x^{2}\right)^{9}\left(15 x^{4}-4 x\right)$

$$
\mathrm{g}^{\prime}=17\left(4 x^{4}+3 x^{2}\right)^{16}\left(16 x^{3}+6 x\right)
$$

Now we can do the same "cross-multiplication" thing that we did in the previous chapter, with the Product Rule. This time, though, + is replaced by - , and there's a denominator.

Our answer becomes:
$\left(f^{\prime} g-f g^{\prime}\right) / g^{2}$

$$
\begin{aligned}
& =\frac{10\left(3 x^{5}-2 x^{2}\right)^{9}\left(15 x^{4}-4 x\right)\left(4 x^{4}+3 x^{2}\right)^{17}-\left(3 x^{5}-2 x^{2}\right)^{10} 17\left(4 x^{4}+3 x^{2}\right)^{16}\left(16 x^{3}+6 x\right)}{\left[\left(4 x^{4}+3 x^{2}\right)^{17}\right]^{2}} \\
& =\frac{10\left(3 x^{5}-2 x^{2}\right)^{9}\left(15 x^{4}-4 x\right)\left(4 x^{4}+3 x^{2}\right)^{17}-17\left(3 x^{5}-2 x^{2}\right)^{10}\left(4 x^{4}+3 x^{2}\right)^{16}\left(16 x^{3}+6 x\right)}{\left(4 x^{4}+3 x^{2}\right)^{34}}
\end{aligned}
$$

All that was done in that last line was (1) simplify the denominator (by using the law of exponents that says what to do with a power of a power) and (2) in the second term of the numerator, rescue the little guy, 17 , from the middle of the muddle and place it at the beginning of the muddle.

Now, we could make like the last section and simplify even more, algebraically. Again, there's just a little bit of repetition (of $4 x^{4}+3 x^{2}$ ), which means cancel. The smallest power of $4 x^{4}+3 x^{2}$ is 16 , so divide both num and denom by $\left(4 x^{4}+3 x^{2}\right)^{16}$, to get:

$$
\frac{10\left(3 x^{5}-2 x^{2}\right)^{9}(15 x-4 x)\left(4 x^{4}+3 x^{2}\right)-17\left(3 x^{5}-2 x^{2}\right)^{10}\left(16 x^{3}+6 x\right)}{\left(4 x^{4}+3 x^{2}\right)^{18}}
$$

Here's a quick way to see why the Quotient Rule holds. (There's a more mathematically rigorous way, but we won't worry about it.)

$$
\begin{gathered}
\frac{d}{d x}(\mathrm{f} / \mathrm{g})=\frac{d}{d x}\left(\mathrm{fg}^{-1}\right), \text { where } \mathrm{g}^{-1} \text { here means } \mathrm{g} \text { to the power }-1 \text { (and NOT the } \\
\text { functional inverse of } \mathrm{g} \text { ) }
\end{gathered}
$$

$=\mathrm{f}^{\prime} \mathrm{g}^{-1}+\mathrm{f}\left(\mathrm{g}^{-1}\right)^{\prime}$, by the Product Rule
$=\mathrm{f}^{\prime} \mathrm{g}^{-1}+\mathrm{f}\left(-\mathrm{g}^{-2}\right) \mathrm{g}^{\prime}$, by the General Power Rule
$=f^{\prime} g^{-1}-\left(f g^{\prime} / g^{2}\right)$, by the definition of negative exponents
$=\left(\mathrm{f}^{\prime} \mathrm{g}-\mathrm{fg} \mathrm{g}^{\prime}\right) / \mathrm{g}^{2}$, by putting both terms over the common denominator $\mathrm{g}^{2}$
In fact, we can diff any quotient by expressing it as a product. Moreover, doing it this way will automatically give us the simplest form of the answer.

Two more Quotient Rule tips conclude this chapter:
(1) Unlike the situation with the Product Rule, the Quotient Rule is not symmetric in the two functions $f$ and $g$. That is, in computing ( $\mathrm{f} / \mathrm{g})^{\prime}$ it matters which is f and which is g .
(2) Don't use the Quotient Rule in cases where it's easier to use the Constant Multiple Rule - for example in diff-ing the function $4 /(x+5)^{2}$. Not all quotients cry out for the Quotient Rule.

Summary so far:
We now know how to diff powers of $x$, linear combos of powers of $x$, powers of such linear combos, and products and quotients of two of such powers. And, remembering the section on "The Case of the Missing Speedometer", we know a little bit about applications of diff-ing.

## Chapter 2 Exercises

## STRAIGHTFORWARD

Diff the following functions:
2.1) $x^{8}$
2.2) $\mathrm{x}^{-8}$
2.3) $x^{1 / 8}$
2.4) $x^{-1 / 8}$
2.5) $2 x^{8}-3 x^{-8}+7 x^{1 / 8}-23 x^{-1 / 8}$
2.6) $\frac{1}{x^{7}}$
2.7) $\sqrt[7]{x}$
2.8) $1+x+x^{2}+x^{3}+x^{4}+x^{5}+x^{6}$
2.9) $x+2 x^{2}+3 x^{3}+4 x^{4}+5 x^{5}+6 x^{6}$
2.10) $x+\sqrt{x}+\sqrt[3]{x}+\sqrt[4]{x}+\sqrt[5]{x}+\sqrt[6]{x}$
2.11) $1+\frac{1}{x}+\frac{1}{x^{2}}+\frac{1}{x^{3}}+\frac{1}{x^{4}}+\frac{1}{x^{5}}+\frac{1}{x^{6}}$
2.12) $x^{1 / 2}+x^{3 / 2}+x^{5 / 2}+x^{7 / 2}$
2.13) $x^{-1 / 2}+x^{-3 / 2}+x^{-5 / 2}+x^{-7 / 2}$
2.14) $x^{1 / 3}+x^{2 / 3}$
$2.15) 1+x^{1 / 4}+x^{1 / 2}+x^{3 / 4}+x+x^{5 / 4}$

If $f(x)=3 x^{2}$ and $g(x)=2 x+1$, find and, if possible, simplify:

$$
\begin{aligned}
& 2.16) f+g \\
& 2.17) f-g \\
& 2.18) 2 f+3 g \\
& 2.19) 5 f g \\
& 2.20) 5 f / g \\
& 2.21) 5 g / f \\
& 2.22) 7 f^{2} \\
& 2.23) 7 g^{2} \\
& 2.24) 7 f^{2} g \\
& 2.25) f^{\prime} \\
& 2.26) \mathrm{f}^{\prime} \mathrm{g} \\
& 2.27) \mathrm{f}^{\prime} / \mathrm{g} \\
& \text { 2.28) } 4 f^{2}+g^{3} \\
& 2.29) f+g+f g \\
& 2.30) \frac{1}{f}+\frac{1}{g} \\
& 2.31) f(g) \\
& 2.32) g(f) \\
& 2.33) g\left(f^{2}\right) \\
& 2.34)[g(f)]^{-2}
\end{aligned}
$$

2.35) If $\mathrm{f}=2 \mathrm{x}$ and $\mathrm{g}(\mathrm{x})=x^{4}-3 x^{2}+7$, find the compositions $\mathrm{f}(\mathrm{g})$ and $\mathrm{g}(\mathrm{f})$.

What are the domain and range of the following functions?
2.36) 100x
2.37) -100x
2.38) $100 \mathrm{x}-50 \mathrm{x}$

$$
\begin{aligned}
& \text { 2.39) } 3 x^{2} \\
& \text { 2.40) } x^{3} \\
& \text { 2.41) } x^{4} \\
& \text { 2.42) } x^{5} \\
& \text { 2.43) } 2 / x \\
& \text { 2.44)(2/x)+1} \\
& \text { 2.45) } 2 /(x+1) \\
& \text { 2.46) } 2 /(x+1)(x-1) \\
& 2.47)(x+1) /) x-1) \\
& 2.48) \frac{2}{(x+1)^{2}(x-1)} \\
& \text { 2.49) } \frac{2}{(x-1)(x-2)(x-3)} \\
& 2.50) \frac{2}{\left.\left.(x-1)^{7}(x-2)^{8}\right) x-3\right)^{9}} \\
& 2.51) \sqrt{x} \\
& 2.52) \sqrt[3]{x} \\
& 2.53) \sqrt[4]{x} \\
& 2.54) \sqrt{x-1} \\
& 2.55) \sqrt[3]{x-1} \\
& 2.56) \sqrt[3]{x+1} \\
& 2.57) \sqrt{(x-1)(x+1} \\
& 2.58) \sqrt[3]{(x-1)(x+1)}
\end{aligned}
$$

Find the inverses of the following functions:
2.59) $\mathrm{x}+23$
2.60) x-23
2.61) 23 x
2.62) $23 \mathrm{x}-4$
2.63) x/23
2.64) $-\mathrm{x} / 23-4$
2.65) $\mathrm{x}^{4}$
2.66) $\sqrt[4]{x}$

Diff the following functions:
2.67) $\left(x^{2}+2 x+1\right)^{4}$
2.68) $5\left(x^{2}+2 x+1\right)^{4}$
2.69) $5\left(x^{2}+2 x+1\right)^{14}$
2.70)5( $\left.4 x^{3}-7 x+2\right)^{14}$
2.71) $\left(4 x^{3}-7 x+2\right)^{14}\left(x^{3}-6 x\right)^{5}$
2.72) $\frac{\left(4 x^{3}-7 x+2\right)^{14}}{\left(x^{3}-6 x\right)^{5}}$
2.73) $\left(12 x^{23}-3 x^{20}+5 x\right)^{19}(3 x+2)^{5}$
2.74) $14 x^{6}\left(12 x^{23}-3 x^{20}+5 x\right)^{19}$
2.75) $\frac{14 x^{6}}{\left(12 x^{23}-3 x^{20}+5 x\right)^{19}}$
2.76) $\frac{\left(12 x^{23}-3 x^{20}+5 x\right)^{19}}{14 x^{6}}$
2.77) $\left(15 x^{3}+2 x^{2}+x+6\right)^{13}\left(42 x^{7}-6 x^{4}+10\right)^{9}$
$2.78)\left(15 x^{3}+2 x^{2}+x+6\right)^{13}\left(42 x^{7}-6 x^{4}+10\right)$
2.79) $\frac{\left(15 x^{3}+2 x^{2}+x+6\right)^{13}}{\left(42 x^{7}-6 x^{4}+10\right)^{9}}$
2.80) $\frac{\left(15 x^{3}+2 x^{2}+x+6\right)^{13}}{\left(42 x^{7}-6 x^{4}+10\right)}$

## APPLICATIONS:

2.81) If, hypothetically, a rocket has traveled a distance of $\left(t^{2}+1\right)^{9}\left(3 t^{3}+4 t^{2}\right)^{10}$ lightyears $t$ seconds after blastoff, at what speed will it be going at that same time?
2.82) If, again hypothetically, a rocket has traveled a distance of $\frac{\left(3 t^{3}+4 t^{2}\right)^{10}}{\left(t^{2}+1\right)^{9}}$ lightsyears $t$ seconds after blastoff, at what speed will it be going at that time?

## CHALLENGERS:

2.83) Find the inverse function to $2 \mathrm{x}^{2}+3$.
2.84) Find the inverse function to -x .
2.85) Find the inverse function to $1 / x$.
2.86) Find the inverse function to $-1 / x$.
2.87) Prove that, for any functions f and g , each of which has an inverse, $[\mathrm{f}(\mathrm{g})]^{-1}=$ $g^{-1}\left(f^{-1}\right)$.
2.88) Prove that, if f has an inverse, then $\mathrm{f}^{-1}$ has an inverse, and in this case we have,

$$
\left(f^{-1}\right)^{-1}=f .
$$

2.89) Give two reasons why the function $f(x)=x^{2}$ doesn't have an inverse. Does $x^{3}$ have an inverse? What about $\mathrm{x}^{4}$ ?
2.90) If we know that a function has an inverse, what can we say about its range?
2.91) For any non-negative integer $n$, find the $n$th deriv of $x^{n}$.
2.92) For any non-negative integer $n$, and for any integer $m>n$, prove that the $m$ th deriv of $\mathrm{x}^{n}$ is 0 .
2.93) For any function $f(x)$, the deriv of the deriv of $f$ is called the second deriv of $f$ (even though a function can't possibly have more than one deriv!). This second deriv is indicated by f ''(x). Similarly, the third deriv of f is defined (if it exists) as the deriv of the second deriv.

For any non-negative integer $n$, compute the following (in terms of $n$ ):
A) $\left(x^{n}\right)^{\prime \prime}$
B) $\left(x^{n}\right), '$,
C) $\left(\mathrm{x}^{n}\right)^{(4)}-$ - that is, the $4^{\text {th }}$ deriv. At 4 and onwards, deriv's are denoted
in that way.
2.94) Generalize: That is, for any integers $m>n$, compute, in terms of $m$ and $n$, the $m$ th deriv of $x^{n}$.
2.95) Referring to the above problem, what is the value of $\left(x^{n}\right)^{(m)}(0)$, when $\mathrm{m}>\mathrm{n}$ ?
2.96) Find a function (or maybe more than one) WHOSE deriv is $x^{2}$. Ditto $x^{3}$.
2.97) In general, for any number $n \neq-1$, find a function whose deriv is $x^{n}$. (The answer will be in terms of $n$.)
2.98) Why doesn't that formula work for $\mathrm{n}=-1$ ? (In a later chapter we'll find a formula which does work for $\mathrm{n}=-1$.)
2.99) A function whose deriv is f is called a primitive or anti-derivative or indefinite integral of f . It turns out that, for any given f , there is more than one anti-derivative of f . But these anti-deriv's are related in some way. What is that way?
2.100) Diff $(x+1)^{7}(x+2)^{8}(x+3)^{9}$.
2.101) Try to find a formula for (fgh)'. (A Product Rule for three functions) What would happen with four functions?
2.102) About the General Power Rule and the Product Rule: What would be a good formula for $\left(g^{n}\right)$ ", where $g=g(x)$ ?
2.103) Find a formula for (fg)'". What about (fg)'" ? Taking a giant leap, what about $(\mathrm{fg})^{(n)}$ ? (If you can do this giant leap, you've discovered Liebniz's Rule.)
2.104) Why do you think the General Power Rule was introduced before the Product Rule?
2.105) Taking $\mathrm{f}=\mathrm{g}$ in the Product Rule, obtain the general Power Rule for $\mathrm{n}=2$.
2.106) Taking $\mathrm{f}=\mathrm{g}^{2}$ in the Product Rule, and using the General Power Rule for $\mathrm{n}=2$, obtain the General Power Rule for $\mathrm{n}=3$. (If we keep this up, we'll get the General Power Rule, from the Product Rule, for all non-negative integers n.)
2.107) Since, as we possibly already know, acceleration means how fast the speed is changing, it isn't terribly surprising that acceleration is found by computing the second deriv of the distance. Knowing this, do the following problem:

A helicopter rises straight up. Its distance, in feet, from the ground after $t$ seconds is given by: $D(t)=t^{2}+1$.
A) How long does it take for the helicopter to rise 20 feet? (No Calc needed)
B) What are the velocity and acceleration when the helicopter is 20 feet above the ground? (Calc needed for this one)
2.108) A toy rocket is tired straight up. Its height, in feet, after $t$ seconds is:

$$
\mathrm{D}(\mathrm{t})=160 \mathrm{t}-16 \mathrm{t}^{2}
$$

A) What is the rocket's velocity at t seconds?
B) What is its velocity at 2 seconds?
C) What is its initial velocity $($ at $\mathrm{t}=0)$ ?
D) What is its acceleration at t seconds?
E) What is its acceleration at 3 seconds?
F) When will it hit the ground again? (Hint: No calc needed for this one, only algebra.)
G) (This part is not straightforward; it previews "maximum and minimum" problems, discussed in a later chapter. You still might have the intuition to solve it.) When will the rocket reach its maximum height? What is that height?
2.109) This is our first Aesop's fable problem: We all know the story of the Hare and the Tortoise, how the fast but over-confident hare lay down for a nap in the middle of the race and didn't wake up in time to run to the finish line before the "slow but steady" tortoise got there first. In our version the Hare doesn't take a nap but is so over-confident, or maybe tired out, that he starts out extremely fast but then - gradually, not suddenly as for a nap - slows down too much, so that the tortoise wins the race.

In terms of functions, the Hare goes, in t hours, a distance:

$$
\mathrm{D}_{H}(\mathrm{t})=\frac{3}{2} t-\frac{1}{8} t^{2} \text { miles }
$$

The Tortoise's distance in t hours is:

$$
\mathrm{D}_{T}(t)=t \text { miles }
$$

A) What is the Hare's velocity after $t$ hours? (Does it keep decreasing, as claimed in the story above?)
B) What is the Hare's initial velocity?
C) What is the Hare's velocity after 1 hour?
D) What is the Tortoise's velocity after $t$ hours? (Is it "slow but steady"?)
E) What is the Tortoise's initial velocity?
F) What is the Tortoise's velocity after an hour?
G) When will the Tortoise start going faster than the Hare?
H) What is the Hare's acceleration at time $t$ ?
I) What is the Tortoise's acceleration at time t ?
J) When will the Tortoise catch up to the Hare?
K) At what speed will the Hare be going at catch-up time?
L) At what speed will the Tortoise be going at catch-up time?
M) What distance, at that time, will they both have traveled?
2.110) Here's another Aesop's fable problem. The fable involved here is less known than The Hare and The Tortoise, and the Tortoise doesn't fare as well in this one. Indeed, at the end, he is unwillingly the very opposite of "slow and steady", and in the wrong direction. (Our problem, though, is shorter and easier than the previous.)

## THE TORTOISE AND THE EAGLE

The Tortoise once upon a time was not the contented fellow that he is today. There was a time when he wished with all his heart that he could fly. As he watched the birds disporting themselves in the clouds he felt sure that if he could get up into the air he could soar with the best of them.

One day he called to an Eagle who was hovering overhead. "Friend Eagle, you are the best flier among all the birds. If you teach me to fly, I will bring you all the treasures of the sea."

The Eagle replied: "But you are asking the impossible, Friend Tortoise. In the first place, you have no wings and, in the second, nature never intended you to fly."

But the Tortoise kept pleading and promising greater and greater rewards. So finally the Eagle said that he would do the best he could.. Telling the Tortoise to hang on, he bore
him high [100 feet, for our problem] into the sky. Then he loosened his hold upon the now thoroughly frightened Tortoise and cried, "All right, start flying."

The poor Tortoise, however, dropped like a plummet and was dashed to pieces on the rocks below.
A) If at $t$ seconds the distance downwards from the top (Remember, 100 feet high) is $D(t)=16 t^{2}$, how long does it take for him to go splat? (No Calc needed)
B) What is his speed when he goes splat?
C) What is his acceleration at splat-time?

## SNEAK PREVIEW

2.111) The definition of deriv leads to a method of approximating roots of numbers which are not perfect powers. Recall the definition:

$$
\mathrm{f}^{\prime}(\mathrm{x})=\lim _{h \rightarrow 0} \frac{f(x+h)-f(x)}{h}
$$

This means that, for small $\mathrm{h}, \frac{f(x+h)-f(x)}{h} \sim \mathrm{f}^{\prime}(\mathrm{x})$, where $\sim$ denotes "equals approximately".
So, using algebra to solve for $\mathrm{f}(\mathrm{x}+\mathrm{h})$, we have the Linear Approximation Formula:

$$
F(x+h) \sim f(x)+h f^{\prime}(x), \text { for small } h .
$$

(So near every point, every function is approximately equal to some linear function.)
Using that formula with $\mathrm{f}(\mathrm{x})=\sqrt{x}, \mathrm{x}=9$, and $\mathrm{h}=1$, approximate $\sqrt{10}$.
2.112) Implicit differentiation means diff-ing when the relation between two variables x and $y$ is given implicitly - for example:

$$
x^{2} y^{3}-4 x y^{5}=3 y .
$$

We see that y is on both sides of the equation. We say that the equation is a relation between x and y - NOT that y is a function of x . (There might be more than one value of $y$, for some values of $x$; $x$ does not necessarily uniquely determine $y$, nor the other way around.)

But we can still sometimes find the deriv y'. We do that by diff-ing both sides of the equation/relation, and then solving for $y$ '. We use both the Product Rule and the General Power Rule. Here's how this goes:

First, using the Product Rule, we have:

$$
\left(\mathrm{x}^{2}\right)^{\prime} \mathrm{y}^{3}+x^{2}\left(y^{3}\right) \mathrm{y}^{\prime}-\left[(4 \mathrm{x})^{\prime} \mathrm{y}^{5}+4 x\left(y^{5} y^{\prime}\right)\right]=(3 \mathrm{y})
$$

Next, using the general Power Rule (on y rather than f), along with some regular diff-ing, we get:

$$
(2 x) y^{3}+x^{2}\left(3 y^{\prime} y^{2}\right)-\left[4 y^{5}+4 x\left(5 y^{\prime} y^{4}\right)\right]=3 y^{\prime}
$$

Using algebra to simplify:

$$
2 x y^{3}+3 x^{2} y^{2} y^{\prime}-4 y^{5}-20 x y^{4} y^{\prime}=3 y^{\prime}
$$

Finally, if we use algebra to solve for y' (NOT y), we get our solution:

$$
\mathrm{y}^{\prime}=\frac{4 y^{5}-2 x y^{3}}{3 x^{2} y^{2}-20 x y^{4}-3}
$$

Notice that the answer (meaning the right side of the equation directly above) does not contain any derivs. Notice also that it contains both x and y , not only x .

Now that you've seen this typical example, use Implicit Diff to find y' when we know that $7 x^{3} y^{4}+3 x y^{2}=12 x y$. (Remember, the answer will contain both x and y .)
2.112: Use Implicit Diff to prove the Power Rule for fractional powers. (Hint: Start with unit fractions: If $\mathrm{y}=\mathrm{x}^{1 / m}$, then $\mathrm{x}=\mathrm{y}^{m}$. Since m is an integer, use the Power Rule for integers. Then, for non-unit fraction, we have $x^{m / n}=\left(x^{(1 / m)}\right)^{n}$ so we can use the General Power Rule.
2.113: Use Implicit Diff, along with the Power Rule for non-negative powers and for the power of -1 , to prove the Power Rule for negative powers.

## CALC LIMERICKS

(Difference quotients, $\frac{f(x+h)-f(x)}{h}$ )
The cellar's a miniscule squatter.
The attic's got something much broader.
Two x's, two f's
to the right and the left.
It helps keep the house in good order.
(A hymn to the $\lim --\lim _{h \rightarrow 0} \frac{f(x+h)-f(x)}{h}$ )
No need to get gloomy and grim concerning that silly ol' lim.
True, the top's pretty beef-y but the bottom part's brief-y
The arrow is even more slim.
(Domain of a function)
Some x's the function can play
And others it can't, nay nay nay.
Take all of the can's and include in our plans. Save the can'ts for a rainy day.
(Range of a function)
f will sometimes attain the y -height.
And sometimes $f$ won't, not quite.
Take all of the yes's
our function expresses.
Save the no's for a rainy night.
(Linear functions)
You can draw a straight line graph pronto.
And what f does it correspond to?
Why, ax + b.
'Tis easy to see unless, of course, you don't want to.
(Slope-intersect form: $\mathrm{y}=\mathrm{mx}+\mathrm{b}, \mathrm{m}=$ slope, $\mathrm{b}=\mathrm{y}$-intercept)
Some statements that might help us cope:
Before x is written the slope
And after the plus
(without too much fuss)
goes the y -intercept (so we hope).
(Point-slope form: $y-y_{1}=m\left(x-x_{1}\right)$ has slope m and passes through the point $\left(x_{1}, y_{1}\right)$ )
The $m$ says how much it doth lean.
And as for the rest of the scene
we've got, just for fun
x -one and y -one
with a comma in between.
(Point-point form: $y-y_{1}=\frac{y_{2}-y_{1}}{x_{2}-x_{1}}\left(x-x_{1}\right)$ passes through both $\left(x_{1}, y_{1}\right)$ and $\left(x_{2}, y_{2}\right)$.)
Two points doth this straight line pass through
( $\mathrm{x}, \mathrm{y}$ ) sub-one and sub-two.
Conversely, these four
(in this math so pure)
determine what's what and who's who.
(Quadratics)
A fair maid from Indianapolis
was an expert on drawing parabolas.
She extended their arms
and showed off their charms
and they looked absolutely fabulous.
(Exponential functions, like $2^{x}: \mathrm{x}$ is in the exponent, not the base)
These guys are as busy as beavers.
They've got Monday morning fever.
They grow, grow, and grow.
They're never zero.
And they don't remain one for long, either.
(Definition of log)
Do logs give you logorrhea?
Is a logjam drawing near?
Just use your credentials
And take exponentials
To make those ol' logs disappear.
(Radians vs. degrees)
A lady named Katy O'Grady
was good at converting to radi-
Ans. "Simple, said she
"just take the degree
"and times it pi over 180."

Her cousin named Mary Magee
knew how to get back to degree.
"Just multiply by
"180 o'er pi.
"My cousin, I'm sure, will agree."
(The Power Rule: $\frac{d}{d x} x^{n}=n x^{n-1}$ )
We can diff any power of $x$
no matter how plain or complex.
n can be miniscule
as an H-molecule
or as large as Tyranno Rex.
Diff-ing x-to-the-n is fun.
Change the n to n minus one.
Then go back to $n$
and use it again
in front of it all, and you're done.
(Power Rule: More Pointers)
In case someone throws a pop quiz
here's how you can think of this biz:
n steps down two ways
for the rest of its days
and x remains right where it is.
(Example of the Power Rule)
It is not my intention to vex.
It is not my intention to hex.
My only intention is merely to mention: deriv of x -square is 2 x .
(General Power Rule: $\frac{d}{d x}[g(x)]^{n}=n[g(x)]^{n-1} g{ }^{\prime}(\mathrm{x})$ )
Are you bored with powers of $x$ ?
What would you like to do next?
Some powers of $g$ will do nicely but watch out for the special effects.
(For example -- )
On, Dasher! On, Dancer! Oh, Prancer!
Don't forget the power in the answer.
It gives it might
and makes it right
also a little fancier.
(Product Rule)
Don't forget - f appears twice.
g will behave likewise.
If you make like a dunce and write them just once -
... well, you'll get what you get for half-price.
(Quotient Rule)
Said a wiseguy named Georgy O'Porgy
"Let's have a Quotient Rule orgy.
"On top, to be nice
"let $g$ appear twice
"and then on the bottom one more g."
--- appeared in Math Horizons
(Chain Rule)
Rinky, dinky, dinky.
g provides the link-y.
But the x and the f
are not to be left
out, unless we run out of ink-y.
(Diff-ing the $\ln$ function: $\frac{d}{d x} \ln x=\frac{1}{x}$ )
Ln, you can't just be.
You haven't to get diff'ed, yesiree.
So give ln the slip
then take the recip
to get one-over-x, easily.
(General $\ln$ Rule: $\frac{d}{d x} \ln g(x)=\mathrm{g}^{\prime}(\mathrm{x}) / \mathrm{g}(\mathrm{x})$ )
On bottom goes the copy.
And what goes on the top-py?
The deriv, g-prime.
It makes it rhyme.
And please don't write it sloppy.
(Diff-ing Trig Functions)

A darling named Clementine said "First sine, then cos, then sine.
"And the minus and plus
"make things even wuss.
"Can't it make up its mind?"

## (Implicit Diff, I)

We're so used to y on one side and we diff it with joy and with pride.
But in this crazy case
y's all over the place and x goes along for the ride.
(Implicit Diff, II)
So we've x's and y's galore.
How can we tackle this chore?
Why, we diff regardless.
Perhaps it seems heartless but mindless it isn't, for sure.
(Implicit Diff, III)
Now, as we proceed down the river diff those terms, sliver by sliver.
Remember, the Chain Rule will be the main rule. but Product Rule isn't chopped liver.
(Curve-sketching)
We can plot and plot 'til we plotz.
But we've got to plot the right spots.
Or those lows and those highs could elude us like flies.
Likewise the flips and the flops.

There was a young man named Kareem who explained, "For a local extreme "the tangent at a
"to rest must lay "and we wish it the pleasant-est dream."

Said his kissin' cousin Trix
"But not vice versa - nix.
"That tangent can flatten
"as low as Manhattan
"with no min, no max, just a mix.

A fair maid from North Minnesota
was drawing a steep asymptot-a.
When it got 'way too high
she murmured bye-bye
and mourned not a single iota.
(l'Hopital's Rule - Good Golly, Miss Molly / here comes l'Hopital-y)
Don't forget - hafta diff twice.
It adds spice to this slice of life.
And it's much more fun
than plain ol' one
and ten-million times more precise.
(l'Hopitals' Rule, II)
There was a fair maid from Nepal.
who dabbled in ol' l'Hopital.
She diff'd 'til she dropped
on bottom and top
then murmured "Nice knowing y'all."
(Exponentials dominate powers. - comparing $\mathrm{b}^{x}$ with $\mathrm{x}^{p}$, as x goes to infinity) No matter how little is $b$ and no matter how big is p $b$ to the x
is the one that out-treks
at least eventually.
(An anti-diff-ing jingle)
We need to be fussy
about that +C .
(Anti-diff-ing the function $\mathrm{f}(\mathrm{x})=\mathrm{x}$ )
It is not my intent to confuse.
It is not my intention to bruise.
My only intention
is merely to mention:
Anti-diff x, get two 2's.
(Anti-diff-ing the squaring function)
It is not my intention to tease.
It is not my intention to tweeze.
My only intention
is merely to mention:
Anti-diff x-square, get two 3's.
(Anti-diff-ing the cubic function)
It is not my intention to force.
It is not my intention to coerce.
My only intention
is merely to mention:
Anti-diff x-cube, get two 4's.
(Power Rule of Anti-Diff-ing: $\int x^{n} d x=\frac{x^{n+1}}{n+1}+C$ )
There's a general rule for all this
(and it's something you don't want to miss):
n moves up and down
all over the town
and x remains right where it is.
$\int x^{-1} d x=\ln x+C$
Minus-one is a cool special case
delicious and dainty as lace.
So don't play the hero.
Don't divide by zero.
If you do, be sure to erase.
(Anti-diff-ing exponentials: $\int e^{k x} d x=\frac{e^{k x}}{k}+C$ )
And now here's a grave admonition
delivered with proper precision:
It's about that k .
It steps down one way
and the x doesn't go where it isn't.
(Anti-diff-ing sin's and cos's: $\int \sin (k x) d x=\frac{-\cos (k x)}{k}+C, \int \cos (k x) d x=\frac{\sin (k x)}{k}+C$ )
A lean lazy lad, name of Jackson
is always forgetting that fraction.
Indeed, he should put
that k underfoot
but he'd rather be busy relaxin'.
(Integration by Parts)
"We need f and g-prime," said Mitch
"and it matters which is which."
"But not to worry," said his cousin Murray.
"It doesn't work out, we'll switch."
(Integration by Substitution)
Root-a-toot toot-a-falutin'.
It's time for some substitutin'.
Take stuff on display
and collapse it away
right along with Leibniz and Newton.
(Substitution jingle)
The differential
is essential.
(Trig
-triggers - i.e., trigonometric substitutions -- $\int \frac{1}{x^{2}+1} d x=\arctan x+C$, using the substitution $\mathrm{x}=\tan \mathrm{t}$ )
Said a frolicking fellow named Nicholas
"Let's begin with something trig-less
"and then trig some trig.
"We'll zag and we'll zig
"to the point of seeming ridiculous."
(A particularly hard problem)
A trig-triggin' trickster from Beacon
is stuck on an odd-powered secant.
An integral table
would render him able
but his conscience is prodding "no peekin'".
(Partial fractions)
Remember, our orgy back when?
Well, we're having an orgy again.
With a different quotient.
We'll take it slow motion.
After all, this is Calc among Friends.
(The $1^{\text {st }}$ Fundamental Theorem of Calculus -- $\int_{a}^{b} f(t) d t=F(b)-F(a)$, where $\mathrm{F}^{\prime}=\mathrm{f}$ )
Don't forget - evaluate twice.
Sorry. but once won't suffice.
Howe'er, the subtraction
is one single action
and I would say that's very nice.
(The $2^{\text {nd }}$ Fundamental Theorem of Calculus -- $\frac{d}{d x} \int_{a}^{x} f(t) d t=f(x)$ )

This is the theorem that gives many functions their anti-deriv's.
So we come away with
more functions to play with
-- great presents for our grandkids.
(Approximate definite integration)
The trapezoid rule can be fun.
All those 2's will get the job done.
But watch out, my friends
for the left and right ends.
At those we will need only 1 .
(Simpson's Rule)
To say it in so many words:
this thing is a matter of thirds.
So nail that ol' Simp
and don't be a wimp.
Dividing by 2's for the birds.
(Solids of revolution)
A strapping young women named Evvie
was handed a solid of rev-y
and asked for the volume.
She answered, quite solemn
"it's not very big but it's heavy."
(Hyperbolic Functions - Are you a hyperbolic-aholic?)
Take the trig I.D's, all kinds add $h$ to those cos's and sin's.
And this derring-do
leaves the diff-ing still true
except for a few minus-signs.
(Improper integrals - infinitely long)
Said a ship-shape chap from the Congo
"Since our region's infinite long-o
"it's likely that you'd
"be inclined to conclude
"that it's infinite big, but you're wrong-o."
---- appeared in Association for Women in Mathematics newsletter
(Improper integrals-infinitely tall)
Said his cousin, dashing Apollo,
"Since our region's infinite tall-o
"it's likely you'll dig
"it be infinite big
"but sorry, that doesn't quite follow."
(Parametric curves)
The vertical line test's a pity.
It stops curves from being pretty.
But now we've got loops
and hulas and hoops
which render us thoroughly giddy.
(Polar coordinates)
With these we can also be arty.
We can have a plotting party.
While away the hours
making petals and flowers
along with Mercenne and Decartes.
(Infinite series)
Said a swingin' young maiden named Brenda
"Since these numbers seem neva $t$ ' end-a,
"it's likely you'd claim
"that their sum does the same
"but you're totally wrong, my friend-a."
(Infinite series - Comparison Test)
Smaller than small is small.
Taller than tall is tall.
And that's how it goes
and that' how one knows
whether anything happens at all.)
(Integral Test: If f decreases and is everywhere positive, then $\int_{1}^{\infty} f(x) d x$ converges if

$$
\text { and only if } \sum_{n=1}^{\infty} f(n) \text { converges ) }
$$

If we know what happens with n's
all x's will follow, my friends.
Yes, the truth emerges:
If one converges
the other will make amends.
(Alternating series, with positive terms whose limit is 0 )
However they rageth and roareth
and wobbleth back and forth,
you'll eventually find
that they make up their mind
some place between south and north.
(MacClaurin series for a paricular function: $\frac{1}{1-r}=1+r+r^{2}+r^{3}+\ldots$ if $-1<\mathrm{r}<1$ ))
The first in our fine repertoire
is one over one-minus-r.
We mustn't forget it
for where we are headed
or else we won't head very far.
An industrious lad from the near East
was summing a long Taylor series.
At the twentieth head
he just shrugged and said
"that's as far as I'm going, my dearies."
(The formula for the coefficient of $\mathrm{x}^{n}$ in the power series for $\mathrm{f}(\mathrm{x}), \frac{f^{(n)}(0)}{n!}$ )
To find it is easy as pie (especially if you try).
There's an exclamation
And a derivation
one low, the other high.

## NOTE ON "THE CASE OF THE MISSING SPEEDOMETER"

When I teach Calculus I, I try to give students, right from the beginning, a sense of what Calculus is. To that end, on the first day of my Calc classes, I cover all or most of this book's Chapter 1 (which motivates and contains the limit definition of derivative without formally defining either "function", "limit", or "derivative" - but by informally defining all three concepts. I begin with the (hypothetical) question, what if you can doesn't have a speedometer? How would we find the speed at, say, 2:00? After giving the "hint" that the car is equipped with an odometer, and we're equipped with a watch, students help me develop the formula which defines "derivative"; they wind up reciting limits (not in so many words) before our very eyes. They also compute the derivatives of functions like 4 x and $5 \mathrm{x}^{2}$. Sometimes we get to $\mathrm{x}^{3}$, and sometimes even the Power Rule.
"You're now doing Calculus," I tell them. "And next class we'll do MORE Calculus."
This "case of the missing speedometer" is, in and of itself, in not new. However, the energy that I give to it seems to be. After defining the derivative $f^{\prime}(x)$ (again, nonrigorously, but with the rigorous notation - lim for "limit", and so on), I point out that " $d$ doesn't have to be distance, and t doesn't have to be time". Meaning, I point out that the idea of derivative can be used to describe and calculate instantaneous change of functions in general. I also "warn" the class that later we'll learn more about functions and limits and at some extent, why.

One course I taught was a section of what my department called "trailer calculus", for students who had previously failed two or three calc courses. At some point during the second or third week I asked the class what the other calc section was doing. I knew, of course, that they were probably learning about functions and limits - as per "Calculus Reform", and I wondered what my students had heard from their friends about this other section.

One student answered, "We don't know WHAT they're doing." I already knew that they knew what WE were doing.

My theory is, functions and limits mean little to students who haven't experienced derivatives. Of course, the rigorous definition of derivative involves both functions and limits, so on first thought there seems to be a vicious cycle. However, again, what my classes do on the first day of calc I is a non-rigorous definition of derivative - "the case of the missing speedometer" as the first example, followed by slight mention of other examples, mostly the idea that there are functions besides displacement of cars, and independent variables besides time (and points besides 2:00).

Another advantage to this approach to the first day of Calculus is, students who have come into the course with Calculus anxiety don't develop additional Calculus anxiety, as they wade through days or weeks of function and limits in prolonged anticipation of the
dreaded derivative. Instead they find themselves actively confronting and computing derivatives, or watching and absorbing as the "class participators" do this computing. Then later, when functions and limits are treated more fully, they're able to understand why, and to feel less alienated.

In his book, What Is Mathematics, Really? (of which the theme is humanism as a philosophy of mathematics), Ruben Hersh writes about how, in a given mathematical theory, there is a difference between the logical sequence of definitions, lemmas, and theorem, and the sequence in which these idea occur to the person researching the theory, as well as the sequence in which the ideas can be taught or written up. Thus, while of course, in order to define the derivative of a function, one needs to first know what "function" and "limit" mean, that is not necessarily how the definition of derivative has to be taught and motivated.

Another thing that I make sure to do, at least in the first couple of weeks of the course, is give continual and recurrent reference to what the derivative of a function means. I keep referring back to "our car with the missing speedometer". For example, after computing the derivative of $\mathrm{x}^{2}$, it's helpful to remind students that this also means that the deriv of $t^{2}$ is $2 t$, and that (hypothetical as this might be) if "our car" has traveled $t^{2}$ miles in $t$ hours, then its speed at $t$ hours is 2 t miles per hour. This brings word problems a mere step away.

I realize, again, that my method is in some sense the opposite of "Calculus reform", but I also believe that it holds to the spirit of Calculus reform. That is, the method, in carefully incorporating the ideas of function and limit into the introduction to derivatives, does take into account that students new to Calculus often aren't very familiar with these ideas.

This approach has worked. "Trailer calculus" students get mostly A's and B's. "This time around I got it," one student told me. Non-trailer calc students also get mostly A's and B's. Of course, it depends on where the course is taught (As an adjunct, I've taught in several places) and on the students that particular semester. "Every class is different," I was advised by a teacher friend, before my very first teaching position. I'm always amazed at how true that is.

Perhaps even more pertinent, one student remarked with a smile, right in class, "Are you sure this is a Calculus class? Are you sure it isn't stand-up comedy?" (I answered, "That's one of the nicest compliments I've ever received.") Until that same student, on another day, again right in class, said, "This is my favorite class this semester." Remember, that was "trailer calculus". And yes, he turned out to receive one of the A's.

